On Invariant Means and Applications to Ergodic Theory and Harmonic Analysis

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by

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ABSTRACT

This thesis is concerned with the existence and properties of invariant means on certain Banach spaces and their applications to ergodic theory and harmonic analysis. The principal results obtained are as follows.

Let G denote either a σ-compact, unimodular amenable group or a countable, cancellative semigroup realized homomorphically by measure preserving transformations on a measure space $(X, \mathcal{S}, \mu)$ via the maps $x \to xg$. Then there exists an increasing sequence $\{S_n\}$ in G such that for all $f \in L_p(X)$, $1 \leqslant p < \infty$ the limit

$$\lim_n |S_n|^{-1} \int_{S_n} f(xg)dg$$

exists in the mean of order p and almost everywhere.

If G is an amenable topological semigroup then it has been shown by H.A. Dye that the ergodic mixing theorem is valid for G. It is proved that the amenability condition can be entirely removed and a mixing theorem is obtained, valid for arbitrary topological semigroups.

The idea of an invariant mean can be dualized to invariant means on the von Neumann algebra of a group.

The existence and in general non-uniqueness of such means is proved. The group von Neumann algebra is also used to show that a classical theorem of Bochner may be rephrased so as to become valid for arbitrary amenable groups rather than Abelian groups.

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INTRODUCTION

This thesis is concerned with the existence and properties of invariant means on certain Banach spaces and their applications to ergodic theory and harmonic analysis.

We begin by giving a somewhat detailed analysis of those results for amenable groups which have direct application to ergodic theory, the most useful of these being results on "summing sequences" and a variant of Day's fixed-point theorem. This latter theorem contains the general abstract mean ergodic theorem which we then obtain as a limit of certain ergodic averages. If in particular the group is σ-compact and unimodular, then the mean ergodic theorem may be expressed in a much more transparent fashion by using summing sequences. A more concrete result may then be obtained by representing the group as measure-preserving transformations on some measure space.

If the mean ergodic theorem is valid for general amenable groups rather than the usual cyclic groups used to generate the classical von Neumann ergodic theorem, what can be said about the individual ergodic theorem? By establishing an intermediate result which somewhat resembles a maximal ergodic theorem, we show that this too depends primarily on amenability and we prove its validity for σ-compact, unimodular groups.

The last ergodic theorem we consider is the ergodic mixing theorem, a result which, as Dye showed, can be expressed in such a way as to render it valid for arbitrary amenable semigroups. We prove (perhaps surprisingly in view of the previous results) that here amenability is not required and that Dye's result may be rephrased so as to be true for any topological semigroup.

The last chapter contains some applications of invariant means to harmonic analysis. We show how the idea of an invariant mean can be "dualized" to invariant means on the von Neumann algebra of a group and prove the existence and in general non-uniqueness of such means. Finally we show that a classical theorem of Bochner may be rephrased so as to become valid for arbitrary amenable groups rather than Abelian groups.

Since it is important from physical considerations that ergodic results should be in terms of semigroups rather than groups, the results of §1-§7 are in general phrased for both semigroups and groups. However, the theory of locally compact semigroups is neither as well-developed nor as complete as the corresponding theory of groups so that we make the following blanket assumption. Except in §8 where the contrary is explicitly stated, all semigroups are assumed <u>countable</u>, <u>discrete</u>, <u>cancellative</u> and with identity.

All results which occur in the literature are accompanied by a reference to the bibliography denoted by [ ]. Any result not carrying such a reference has not been found in the literature.

NOTATION

All groups considered here are locally compact, Hausdorff topological groups denoted by G and are multiplicatively written with identity element e. The left Haar measure of a (Borel) measurable set A is written $|A|$ and the characteristic function of A is written χ_A. The differential of left Haar measure is denoted by dg and the Radon-Nikodym derivative of right Haar measure with respect to left Haar measure by $\Delta(g)$. Hence

$$\int_G \Delta(g^{-1})f(g^{-1})dg = \int_G f(g)dg$$

for any (left) integrable function f.

By $C_{00}(G)$ we denote the linear space of continuous functions on G vanishing outside compact sets and by $C_0(G)$ [resp. $CB(G)$] the Banach space, with the uniform norm, of all continuous functions on G which vanish at infinity [resp. which are bounded].

For any p, $1 \leqslant p < \infty$, we denote by L_p or $L_p(G)$ the Banach space of measurable functions f which are p-th power integrable and with the norm $||f||_p$. The space $L_\infty(G)$ consists of all essentially bounded, measurable functions with norm $||\cdot||_\infty$. If the meaning is unambiguous we may simply write $||\cdot||$. If X is any of the above function spaces, we denote by X^r the real subspace of real-valued functions. We also denote by F the set of all weight functions (= non-negative, integrable functions f with $||f||_1 = 1$) on G.

We consider also the space M(G) of bounded Radon measures on G and $M_1^+(G)$ the subset of all positive measures μ with $||\mu|| = 1$. By δ_g, $g \in G$, we denote the measure concentrated on g and with total mass 1.

For any function $x \in L_p(G)$, $1 \leq p < \infty$ and any $\mu \in M(G)$ we denote by x^*, $\check{x}$, $\tilde{x}$ and μ^*, $\tilde{\mu}$ the functions and measures

$$x^*(g) = \Delta(g^{-1})\overline{x(g^{-1})}, \quad \check{x}(g) = x(g^{-1}), \quad \tilde{x}(g) = \overline{x(g^{-1})}$$

where $\bar{}$ denotes complex conjugation.

$$\mu^*(E) = \overline{\mu(E^{-1})}, \quad \tilde{\mu}(E) = \overline{\int_{E^{-1}} \Delta(g^{-1})d\mu(g)}$$

For a function $f \in L_p(G)$, $1 \leq p < \infty$ and a measure $\mu \in M(G)$, we define the convolutions

$$f*\mu(g) = \int_G f(gh^{-1})\Delta(h^{-1})d\mu(h)$$

$$\mu*f(g) = \int_G f(h^{-1}g)d\mu(h)$$

Under convolution and the * operation, $L_1(G)$ and M(G) are Banach * algebras. Note also that $\mu*f \in L_p(G)$ and that $L_1(G)$ may be identified with the (two-sided) ideal in $M(G)$ of all measures, absolutely continuous with respect to left Haar measure.

We also have

$$f*\tilde{\delta}_g = f_g \quad \text{where} \quad f_g(h) = f(hg)$$

and $$\delta_{g^{-1}}*f = {}_gf \quad \text{where} \quad {}_gf(h) = f(gh)$$

If G is a semigroup then most of the above applies. We generally write $\ell_p(G)$ for $L_p(G)$ and note that convolution is now defined by

$$x*y(g) = \sum_{hk=g} x(h)y(k)$$

Although functions such as $\check{x}$, $\tilde{x}$ are in general not defined we may still consider functions of the form $f*\check{x}$ by defining

$$f*\check{x}(g) = \sum_{h\in G} f(gh)x(h)$$

which is consistent with the group definition.

Finally if X is a Banach space with dual space X^* we sometimes denote the duality by $\langle x^*,x\rangle$ rather than $x^*(x)$.

CHAPTER I

AMENABLE GROUPS AND SEMIGROUPS

In this chapter we consider an important class of groups and semigroups, namely, those which are amenable. We do not give an exhaustive study of these structures but restrict ourselves to developing mainly those ideas and properties which have application to ergodic theory. We commence (§1) by defining amenability in terms of the existence of invariant means on $L_\infty(G)$ and consider Hulanicki's alternative approach via topologically invariant means. (Here we prove that these ideas of topological invariance and invariance coincide.) In §2 we consider amenability as a limiting process on finite means and use this approach to give some examples of amenable groups. Følner's conditions and the subsequent idea of summing sequences are introduced in §3. In §4 we consider in some detail the concept of almost convergence, in particular we look at a recent result due to Douglass and obtain a suitable generalization for our purposes. Finally §5 is devoted to a study of Day's fixed-point theorem, a vital result for later application to ergodic theory.

Throughout, definitions and results are in general given for groups, the semigroup case being either analogous or treated separately.

§1. MEANS AND AMENABILITY

Definition 1.1. A linear functional m on $L_\infty(G)$ is called a mean on $L_\infty(G)$ if it satisfies the following properties

(i) $m(\bar{f}) = \overline{m(f)}$;

(ii) $f \geqslant 0 \Rightarrow m(f) \geqslant 0$;

(iii) $m(1) = 1$, where 1 denotes the constant function.

Note that (i) actually follows from (ii) and that a mean m is necessarily bounded with $||m|| = 1$ and satisfies

$$\operatorname*{ess\,inf}_{g \in G} f(g) \leqslant m(f) \leqslant \operatorname*{ess\,sup}_{g \in G} f(g) \quad \text{for all real } f$$

From the set of means we single out an important subset.

Definition 1.2. A mean m is called finite if $m(f) = \int_G \alpha(g)f(g)dg$ where α is a weight on G, i.e. α is a non-negative function in $L_1(G)$ with $||\alpha||_1 = 1$.

We generally identify the set of finite means with the set of weight functions and denote either set by F. The importance of finite means (quite apart from their simple structure) stems from

Proposition 1.3. ([25] p.94). The set of finite means is w*-dense in the set of all means on $L_\infty(G)$.

Proof. Using a general result on vector lattices (e.g. [32] p.16) there exists a net $\{x_\gamma\}$ of non-negative functions in $L_1(G)$ which is w*-convergent to m. Since $m(1) = 1$ we may (normalizing if necessary) assume that $||x_\gamma||_1 = 1$ and the result follows.

A useful and immediate property of the set of means is the following.

Proposition 1.4. ([25] p.92). The set of means on $L_\infty(G)$ is convex and w*-compact.

Definition 1.5. A mean m is called left (resp. right) invariant if $m({}_gf) = m(f)$ (resp. $m(f_g) = m(f)$) for all $f \in L_\infty(G)$ and $g \in G$. m is called invariant if it is both left and right invariant.

For groups the following result due to Day [3] is important.

Theorem 1.6. Let G be a group and suppose that there exists a left invariant mean on $L_\infty(G)$. Then there exists a (two-sided) invariant mean on $L_\infty(G)$.

Proof. (Adapted from [23] p.233-4). Let m be a left invariant mean. We show firstly the existence of a right invariant mean. In fact if we define

$$m'(f) = m(\check{f})$$

(recall that $\check{f}(g) = f(g^{-1})$) then it is easily seen that m' is a mean and also

$$m'(f_g) = m[(f_g)^{\vee}] = m[{}_{g^{-1}}(\check{f})] = m(\check{f}) = m'(f)$$

Now for $f \in L_\infty(G)$, define $f'(g) = m'({}_g f)$. Continuity of m' shows that $f' \in L_\infty(G)$ and the map $f \to f'$ is linear. Write $m_0(f) = m(f')$. We claim that m_0 is an invariant mean.

Since $f \to f'$ is linear, m_0 is a linear functional on $L_\infty(G)$ and the fact that m and m' are means quickly show that m_0 is also a mean. Now for $g, h \in G$ we have

$$({}_g f)'(h) = m'({}_h({}_g f)) = m'({}_{gh} f) = f'(gh) = {}_g(f')(h).$$

Hence

$$m_0({}_g f) = m(({}_g f)') = m({}_g(f')) = m(f') = m_0(f)$$

so that m_0 is left invariant. Also,

$$(f_g)'(h) = m'({}_h(f_g)) = m'(({}_h f)_g) = m'({}_h f) = f'(h)$$

by right invariance of m'. Hence

$$m_0(f_g) = m((f_g)') = m(f') = m_0(f)$$

and m_0 is an invariant mean.

Remark. The second part of the proof is valid for semigroups as well as for groups and shows that if G is a semigroup with left and right invariant means on $\ell_\infty(G)$ then there exists an invariant mean. However for semigroups it is not clear when the existence of a left invariant mean implies the existence of a right invariant mean.

Definition 1.7. A group G is called amenable if there exists a left invariant mean on $L_\infty(G)$.

By theorem 1.6 we do not need to distinguish between left and right amenability. For semigroups, however, we do.

Definition 1.8. A semigroup G is called left (resp. right) amenable if there exists a left (resp. right) invariant mean on $\ell_\infty(G)$. G is called amenable if it is both left and right amenable or what is equivalent if there exists a (two-sided) invariant mean on $\ell_\infty(G)$.

If G is the semigroup of non-negative integers under addition, $\ell_\infty(G)$ is then the space of all bounded sequences. The existence of "Banach Limits" (see e.g. [35] p.58) then ensures that G is amenable. This semigroup (for the purpose of ergodic theory at least) may be regarded as the fundamental amenable semigroup and many results will be couched in such terms as to make the generalization from this to more general (semi) groups more transparent. Another well-known example is given in the following

Proposition 1.9. Every compact group or finite semigroup is amenable.

Proof. If G is a compact group, $L_\infty(G) \subset L_1(G)$ and the Haar integral is the required invariant mean. More specifically, we define m by

$$m(f) = \int_G f(g)dg \quad \text{for } f \in L_\infty(G).$$

If G is a finite semigroup, then being cancellative by hypothesis, G is a group and the above proof applies.

In [25], Hulanicki introduced an apparently different concept of invariance. His idea is as follows

Definition 1.10. A mean m on $L_\infty(G)$ is called topologically left (resp. right) invariant if for any weight $\alpha \in F$ and $f \in L_\infty(G)$,

$$m(\alpha * f) = m(f) \quad (\text{resp. } m(f * \tilde{\alpha}) = m(f))$$

m is called topologically invariant if

$$m(\alpha * f * \tilde{\beta}) = m(f) \quad \text{for all } \alpha,\ \beta \in F,\ f \in L_\infty(G)$$

Note. Although this definition makes sense only if G is a group since in the semigroup case $\tilde{\alpha}$ is not defined, we can extend the definition to semigroups by defining

$$f*\tilde{\alpha}(g) = \sum_{h \in G} f(gh)\overline{\alpha(h)}$$

Also if G is discrete then it is readily seen that a mean m is left (resp. right) topologically invariant iff m is left (resp. right) invariant. The two concepts of invariance and topological invariance have led to quite a large number of results concerning invariant means being proved again for topologically invariant means (witness for example the results of Hulanicki [25] and Day [4]). Hulanicki showed that every topologically invariant mean is an invariant mean and later Namioka [34] showed that the existence of an invariant mean implies the existence of a topologically invariant mean. Our first major task is to prove the actual equivalence of these two ideas.

Lemma 1.11. ([25] p.92). A mean m is topologically left (resp. right) invariant or topologically invariant iff for any $f \in L_\infty(G)$ and $\mu, \nu \in M_1^+(G)$ we have

$$m(\mu * f) = m(f) \quad (\text{resp. } m(f*\tilde{\nu}) = m(f))$$

$$\text{or} \quad m(\mu * f * \tilde{\nu}) = m(f).$$

Proof. We consider the topologically invariant case, the left or right cases being similar. If m is topologically invariant, μ, $\nu \in M_1^+(G)$ and $\alpha \in F$ then $\alpha*\mu$, $\alpha*\nu \in F$ and

$$\begin{aligned} m(\mu*f*\tilde{\nu}) &= m(\alpha*(\mu*f*\tilde{\nu})*\tilde{\alpha}) \\ &= m((\alpha*\mu)*f*(\alpha*\nu)^{\sim}) \\ &= m(f). \end{aligned}$$

Since $F \subset M_1^+(G)$ the converse is trivial. Note incidently that this lemma shows that m is topologically invariant iff m is both topologically left and rightinvariant.

<u>Corollary 1.12</u>. ([25] p.93). If m is topologically left (resp. right) invariant then m is left (resp. right) invariant.

Proof. By the note above we need only consider the non-discrete i.e. group case. But then for all $g \in G$, $f \in L_\infty(G)$ we have respectively,

$$m({}_g f) = (\delta_{g^{-1}}*f) = m(f) \quad \text{and}$$

$$m(f_g) = m(f*\tilde{\delta}_g) = m(f)$$

The converse proposition e.g. that every left invariant mean is topologically left invariant is more delicate. The proof depends largely on deriving a form of Egoroff's theorem valid for <u>nets</u> rather than sequences.

<u>Theorem 1.13</u>. Let m be a left invariant mean on $L_\infty(G)$. Then m is topologically left invariant. Similarly with right replacing left.

For $f \in L_\infty(G)$, $x \in L_1(G)$, left invariance of m gives

$$m(({}_g x)*f) = m({}_g(x*f)) = m(x*f) \quad \text{for all } g \in G.$$

Hence $x \to m\,(x*f)$ defines a left invariant bounded linear functional on $L_1(G)$ so that by uniqueness of Haar measure there exists a constant $k(f)$ such that

$$m(x*f) = k(f) \int_G x(g)dg \quad \text{for all } x \in L_1(G).$$

It is immediate that k is a mean on $L_\infty(G)$. Further if $x \in F$ then $x*x \in F$ so that

$$k(x*f) = m(x*(x*f)) = m((x*x)*f) = k(f)$$

and k is <u>topologically</u> left invariant. The theorem will be proved once we show that m = k.

Fix $f \in L_\infty(G)$. Choose a net $\{x_\gamma\}_{\gamma \in \Omega} \subset F$ such that $w^*\text{-}\lim x_\gamma = m$ and define F_γ by

$$F_\gamma(g) = \langle {}_g x_\gamma, f \rangle$$

left invariance of m means that $F_\gamma \to m(f)$ pointwise on G. The theorem will follow from the following

Lemma 1.14. $F_\gamma \to m(f)$ almost uniformly on every compact subset of G.

Proof. If we were dealing with sequences rather than nets then the lemma would be a trivial application of Egoroff's theorem. With nets, however, a little delicacy is required.

Let K be a compact set with $|K| > 0$. For k a positive integer, $\gamma \in \Omega$ define

$$E_{k,\gamma} = \bigcap_{\gamma' \geq \gamma} \{g \in K : |F_{\gamma}(g) - m(f)| \leq 1/k\}.$$

Since F_γ is continuous, $E_{k,\gamma}$ is a compact subset of K. Note that for fixed k, $\{E_{k,\gamma}\}$ is an increasing net with $\bigcup_\gamma E_{k,\gamma} = K$. Let χ_K, $\chi_{E_{k,\gamma}}$ denote the characteristic functions of K and $E_{k,\gamma}$ respectively. We then have that $\{\chi_{E_{k,\gamma}}\}$ is a bounded monotone increasing net in $L_\infty(G)$ for each k and $\chi_K = \sup_\gamma \chi_{E_{k,\gamma}}$. Now $L_\infty(K)$ may be regarded as a von Neumann algebra on the Hilbert space $L_2(K)$. The predual of $L_\infty(K)$ is $L_1(K)$ so that every non-negative element in $L_1(K)$ is a normal positive linear functional on $L_\infty(K)$ (see [7], chapitre 1, §3, §4). Hence $\langle \chi_K, \chi_K \rangle = \sup_\gamma \langle \chi_K, \chi_{E_{k,\gamma}} \rangle$ or

$$\lim |E_{k,\gamma}| = |K| \quad \text{for each } k.$$

Fix $\varepsilon > 0$. For each k, choose γ_k such that $|K\backslash E_{k,\gamma_k}| < \varepsilon/2^k$ and let $E_0 = \bigcap_k E_{k,\gamma_k}$. E_0 is a compact set and

$$|K\backslash E_0| = |\bigcup_{k=1}^{\infty} K\backslash E_{k,\gamma_k}| \leqslant \sum_{k=1}^{\infty} |K\backslash E_{k,\gamma_k}| < \varepsilon$$

Finally it is clear that $F_\gamma \to m(f)$ uniformly on E_0.

Proof of Theorem 1.13. By the above lemma we can find a compact set E with $|E| > 0$ such that $F_\gamma \to m(f)$ uniformly on E. Therefore

$$\lim \int_E F_\gamma(g)dg = m(f)|E|. \tag{1}$$

But
$$\begin{aligned}\int_E F_\gamma(g)dg &= \int_G \chi_E(g)\,[\int_G x_\gamma(g^{-1}j)f(h)dh]\,dg \\ &= \int_G (\chi_E * x_\gamma)(h)f(h)dh \\ &= \langle \chi_E * x_\gamma,\ f\rangle \\ &= \langle x_\gamma,\ \chi_E^* * f\rangle\end{aligned}$$

so that

$$\lim \int_E F_\gamma(g)dg = m(\chi_E^* * f)$$

Hence by (1),

$$\begin{aligned}m(f)\ |E| &= m(\chi_E^* * f) \\ &= k(f) \int_G \chi_E^*(g)dg \\ &= k(f)\ |E|.\end{aligned}$$

Therefore $m(f) = k(f)$ and f being arbitrary, $m = k$. Hence m is a topologically left invariant mean.

§2 WEAK AND STRONG INVARIANCE

Since the set F of finite means on $L_\infty(G)$ is w*-dense in the set of all means, the existence of invariant means should clearly be connected with the existence of nets of finite means which in some sense "converge to invariance". This useful alternative approach to amenability was first introduced by Day [4] for the discrete case and by Hulanicki [25] for the general locally compact case.

Definition 2.1. A net $\{\alpha_\gamma\}$ of finite means is w*-[norm] convergent to left (right) invariance if for any weight α,

$$\text{w*-lim } (\alpha * \alpha_\gamma - \alpha_\gamma) = 0 \ [\lim ||\alpha * \alpha_\gamma - \alpha_\gamma||_1 = 0]$$

$$(\text{w*-lim } (\alpha_\gamma * \alpha - \alpha_\gamma) = 0 \ [\lim ||\alpha_\gamma * \alpha - \alpha_\gamma||_1 = 0])$$

$\{\alpha_\gamma\}$ is w*-[norm] convergent to invariance if for all weights α, β,

$$\text{w*-lim } (\alpha * \alpha_\gamma * \beta - \alpha_\gamma) = 0 \ [\lim ||\alpha * \alpha_\gamma * \beta - \alpha_\gamma||_1 = 0].$$

We note at once that if $\{\alpha_\gamma\}$ is w*-(norm) convergent to left invariance, then $\{\alpha_\gamma^*\}$ is w*-(norm) convergent to right invariance (this for groups only). The connection between the existence of such nets (in the w*-case at least) and amenability is immediate.

<u>Theorem 2.2</u>. ([25] p.94). There exists a left (right, two-sided) invariant mean on $L_\infty(G)$ if and only if there exists a net of finite means, w*-convergent to left (right, two-sided) invariance.

Proof. We treat the left case only. Let m be left invariant. By proposition 1.3 there exists a net $\{\alpha_\gamma\}$ of finite means, w*-convergent to m. Hence for any weight α and $f \in L_\infty(G)$ we have

$$\begin{aligned} 0 = m(\alpha * f - f) &= \lim \langle \alpha_\gamma, \alpha^* * f - f\rangle \\ &= \lim (\langle \alpha_\gamma, \alpha^* * f\rangle - \langle \alpha_\gamma, f\rangle) \\ &= \lim (\langle \alpha * \alpha_\gamma, f\rangle - \langle \alpha_\gamma, f\rangle) \\ &= \lim \langle \alpha * \alpha_\gamma - \alpha_\gamma, f\rangle \ , \end{aligned}$$

so that $\{\alpha_\gamma\}$ is w*-convergent to left invariance. Conversely if $\{\alpha_\gamma\}$ is any such net then by proposition 1.3, there exists a mean m which is an accumulation point of the net $\{\alpha_\gamma\}$. The equations above, read in reverse order and passing to a subnet if necessary, now show that m is left invariant.

It is not immediately clear from the definition that convergence to invariance coincides with convergence to both left and right invariance. We will prove this fact which will be of some use later. Firstly we strengthen somewhat the conditions for convergence.

<u>Lemma 2.3</u>. A net $\{\alpha_\gamma\}$ of finite means is w*-[norm] convergent to left (right, two-sided) invariance if and only if for all $\mu, \nu \in M_1^+(G)$,

$$w^*\text{-}\lim\,(\mu*\alpha_\gamma - \alpha_\gamma) = 0 \quad [\lim\,||\mu*\alpha_\gamma - \alpha_\gamma||_1 = 0]$$

$$(w^*\text{-}\lim\,(\alpha_\gamma*\mu - \alpha_\gamma) = 0 \quad [\lim\,||\alpha_\gamma*\mu - \alpha_\gamma||_1 = 0],$$

$$w^*\text{-}\lim\,(\mu*\alpha_\gamma*\nu - \alpha_\gamma) = 0 \quad [\lim\,||\mu*\alpha_\gamma*\nu - \alpha_\gamma||_1 = 0])$$

Proof. We treat the left case only. Note that since $F \subseteq M_1^+(G)$ the "if" part is trivial.

Suppose then that $\{\alpha_\gamma\}$ is w*-convergent to left invariance. Let $\mu \in M_1^+(G)$, $f \in L_\infty(G)$. Then of $\alpha \in F$ we have

$$\begin{aligned} 0 &= \lim\,\langle \alpha*\alpha_\gamma - \alpha_\gamma, \mu^* * f\rangle \\ &= \lim\,\langle \mu*\alpha*\alpha_\gamma - \mu*\alpha_\gamma, f\rangle. \end{aligned}$$

On the other hand, since $\mu*\alpha \in F$,

$$0 = \lim\,\langle \mu*\alpha*\alpha_\gamma - \alpha_\gamma, f\rangle$$

and these last two equations show that

$$\lim\,\langle \mu*\alpha_\gamma - \alpha_\gamma, f\rangle = 0.$$

A similar argument proves the norm case.

The following result will also be needed.

Lemma 2.4. If $\{\alpha_\gamma\}$ is w*-[norm] convergent to left (right) invariance then so is $\mu*\alpha_\gamma$ $(\alpha_\gamma*\mu)$ for all $\mu\in M_1^+(G)$.

Proof. Suppose that $\{\alpha_\gamma\}$ is w*-convergent to left invariance (the other three cases being similar). Let $\mu\in M_1^+(G)$. Then for all $\alpha\in F$, we have

$$\begin{aligned} &\text{w*-lim } [\alpha*(\mu*\alpha_\gamma) - \mu*\alpha_\gamma] \\ &= \text{w*-lim } [(\alpha*\mu)*\alpha_\gamma - \alpha_\gamma] - \text{w*-lim } [\mu*\alpha_\gamma - \alpha_\gamma] \\ &= 0 \text{ by lemma 2.3.} \end{aligned}$$

As a corollary we can now prove

Proposition 2.5. A net $\{\alpha_\gamma\}$ of finite means is w*-(norm) convergent to invariance if and only if it is w*-(norm) convergent to both left and right invariance.

Proof (for the w*-case). Suppose that $\{\alpha_\gamma\}$ is w*-convergent to invariance. By lemma 2.3. we have for all $\mu,\ \nu\in M_1^+(G)$, w*-lim $(\mu*\alpha_\gamma*\nu-\alpha_\gamma) = 0$.
Putting firstly $\mu = \delta_e$ and ν a weight α, and secondly $\mu = \alpha$ and $\nu = \delta_e$ we obtain

$$\text{w*-lim } (\alpha_\gamma*\alpha - \alpha_\gamma) = \text{w*-lim } (\alpha*\alpha_\gamma - \alpha_\gamma) = 0.$$

If conversely $\{\alpha_\gamma\}$ is w*-convergent to both left and right invariance, then for all α, $\beta \in F$ we have

$$\text{w*-lim } (\alpha * \alpha_\gamma * \beta - \alpha_\gamma)$$
$$= \text{w*-lim } [(\alpha * \alpha_\gamma) * \beta - (\alpha * \alpha_\gamma)] + \text{w*-lim } [\alpha * \alpha_\gamma - \alpha_\gamma].$$

The second limit is zero and by lemma 2.4 so is the first. The result now follows.

It is obvious that if there exists a net $\{\alpha_\gamma\}$ of finite means norm convergent to left invariance then there exists a net $\{\alpha'_\gamma\}$ of means w*-convergent to left invariance (take e.g. $\alpha'_\gamma = \alpha_\gamma$). It is perhaps surprising that the converse is also true. In the discrete case this was proved by Day [4]. Namioka ([33], theorem 2.2) then found a very elegant proof of this fact which extends at once to the general case, as noted by Hulanicki ([25] p.94).

<u>Proposition 2.6</u>. If $\{\alpha_\gamma\}$ is a net w*-convergent to left invariance then there exists also a net $\{\alpha_\gamma\}$ norm convergent to left invariance, with similar results for right and two-sided invariance.

Proof. (Namioka [33], proof of theorem 2.2). Let E denote the product $[L_1(G)]^F$. E is a locally convex linear topological space under the product of norm topologies. Define the linear map $T:L_1(G) \to E$ as follows. For $x \in L_1(G)$ and $\alpha \in F$ define

$$[Tx](\alpha) = \alpha * x - \alpha.$$

Now the weak topology on E coincides with the product of the weak topologies ([27] p.160) and since

$$\text{w-lim}_{\gamma} (\alpha * \alpha_\gamma - \alpha_\gamma) = 0 \quad \text{for all } \alpha \in F,$$

we have that 0 is in the weak closure of T(F). Since T(F) is convex, the weak closure of T(F) is the same as the closure $[T(F)]^-$ of T(F) relative to the topology on E (see e.g.[14] p.119). Hence $0 \in [T(F)]^-$ which means that there is a net $\{\alpha'_\gamma\}$ of weights such that for all $\alpha \in F$,

$$\lim_{\gamma} ||\alpha * \alpha'_\gamma - \alpha'_\gamma|| = 0.$$

This approach to amenability via nets allows us to describe two further examples of amenable (semi)-groups. Call a group locally finite if every compact subset generates a compact subgroup and similarly for semi-groups. Then we have

Theorem 2.7. (i) Every locally finite (semi)-group is amenable and (ii) Every Abelian (semi)-group is amenable.

Proof. (i) This result is due to Day ([4] p.517). The proof we give is quite different.

Note firstly that if G is a locally finite semigroup then being cancellative, G is actually a group. So it suffices to consider the locally compact group case. We construct a net of finite means which is norm convergent to left invariance.

For K a compact subgroup with non-empty interior, let $\alpha_K = |K|^{-1}\chi_K$. $\{\alpha_K\}$ is then a net of weight functions with the partial order defined by $K \supset K'$. If α is any weight on G we have

$$\begin{aligned}
||\alpha * \alpha_K - \alpha_K||_1 &= \int_G \left| \int_G \alpha(h)\alpha_K(h^{-1}g)dh - \alpha_K(g)\right| dg \\
&\leq \int_G \int_G \alpha(h)\,|\alpha_K(h^{-1}g) - \alpha_K(g)|\,dgdh \\
&= |K|^{-1}\int_G \int_G \alpha(h)\,|\chi_{hK}(g) - \chi_K(g)|\,dgdh \\
&= |K|^{-1}\int_G \alpha(h) \int_G \chi_{hK \Delta K}(g)dgdh \\
&= |K|^{-1}\int_G \alpha(h)\,|hK \Delta K|\,dh.
\end{aligned}$$

Now fix $\varepsilon > 0$ and choose a compact set Q such that $\int_Q \alpha(h)dh > 1 - \varepsilon/2$. Choose K a compact subgroup such that $Q \subset K$. Then,

$$\begin{aligned}
||\alpha * \alpha_K - \alpha_K||_1 &\leq |K|^{-1} \int_Q \alpha(h)\,|hK \Delta K|\,dh \\
&\quad + |K|^{-1} \int_{G \setminus Q} \alpha(h)\,|hK \Delta K|\,dh \\
&\leq 0 + |K|^{-1} 2|K|\ \varepsilon/2 = \varepsilon.
\end{aligned}$$

(ii) ([25] p.101). Assume now that G is an Abelian semigroup (the proof includes the group case). For any finite subset $\delta = \{\alpha_1, \ldots, \alpha_k\} \subset F$ and positive integer n, let $\gamma = (\delta, n)$. Partially order the pairs (δ, n) by $(\delta', n') \geqslant (\delta, n)$ if $\delta' \supset \delta$ and $n' \geqslant n$.

Let

$$\alpha_\gamma = n^{-k} \sum_{0<j_1,\ldots,j_k\leqslant n} \alpha_1^{j_1} * \ldots * \alpha_k^{j_k}$$

(where α^j denotes $\alpha * \ldots * \alpha$ (j times)). If now α is a weight and if $\alpha = \alpha_i \in \delta$ and n is a positive integer then for $\gamma = (\delta, n)$ we have

$$\begin{aligned} ||\alpha*\alpha_\gamma - \alpha_\gamma|| &= n^{-k} || \sum_{0<j_1,\ldots,j_k\leqslant n} \alpha_1^{j_1} * \ldots * \alpha_i^{j_i} * \ldots * \alpha_k^{j_k}(\alpha_i^{j_i+1} - \alpha_i^{j_i})||_1 \\ &= n^{-1} || \sum_{j=1}^{n} (\alpha_i^{j+1} - \alpha_i^{j})|| \\ &\leqslant 2n^{-1} \end{aligned}$$

letting $n \to \infty$ with γ, the result follows.

§3 FØLNER CONDITIONS AND SUMMING SEQUENCES

If we look back at the various examples of amenable groups given above, we see that in a certain rough sense, compact groups are the "most" amenable followed by locally finite groups. The reason in each case seems to be that compact sets generate subgroups which are not too large. The Abelian case makes this idea a little clearer. Namely, the commutative law forces the growth of a sequence $\{K^n\}$ (K compact) to be of low order. In this spirit consider the following two conditions on a (semi)-group G.

(FC) (Følner Condition). If $\varepsilon > 0$ and compact set $K \subset G$ are given, there is a compact set U with $0 < |U| < \infty$ such that $|gU \Delta U| < \varepsilon\ |U|$ for all $g \in K$.

(A) If $\varepsilon > 0$ and compact set $K \subset G$ are given ($K \neq \phi$), there is a compact set U with $0 < |U| < \infty$ such that $|KU \Delta U| < \varepsilon |U|$.

It is trivial that both of these conditions are satisfied if G is locally finite - take U to be the compact subgroup generated by K. If (P) denotes either of the conditions (FC) or (A) then it is occasionally of some importance to know how such sets U are located in G

We may for example ask whether the following localization result is true.

(P_{loc}) Let (ε, K) be given as in (P) together with any other compact set $E \subset G$. Then there is a compact set U satisfying (P) for (ε, K) such that $E \subset U$.

The following theorem is decisive

<u>Theorem 3.1.</u> If G is a group then (amenable) $\Leftrightarrow$ (FC) $\Leftrightarrow$ (A). Further if (P) denotes either (FC) or (A) then (P) $\Leftrightarrow$ (P_{loc}).

This result is as useful as it is deep. It was first proved in the discrete case by Følner [18]. The general locally compact case was settled by Emerson and Greenleaf [15]. It is interesting to compare the two proofs; the general case is surprisingly so much more difficult. For this reason we omit the proof. It suffices to say that the general idea is to use the net approach to amenability and to show that the weight functions may be chosen to be normalized characteristic functions.

For semigroups the corresponding results were obtained by Namioka [33] and read as follows

Theorem 3.2. Let G be a semigroup. The following conditions are equivalent

(i) G is amenable

(ii) (FC') If $\varepsilon > 0$ and finite set $K \subset G$ are given there is a finite set U such that $|gU \Delta U| < \varepsilon|U|$ and $|Ug \Delta U| < \varepsilon|U|$ for all $g \in K$

(iii) (A') If $\varepsilon > 0$ and non-empty finite set K are given there is a finite set U such that $|KU \Delta U| < \varepsilon|U|$ and $|UK \Delta U| < \varepsilon|U|$.

We may similarly define the condition (P'_{loc}) where P' denotes either (A') or (FC') and obtain the equivalence of (i), (ii) (iii) with (P'_{loc}).

Note that in the semigroup case, the conditions (FC') and (A') appear in a two-sided form whereas for groups the corresponding conditions (FC) and (A) do not. This is due to the fact that our semigroups are discrete and hence in a sense unimodular, a condition which we do not in general require for groups. However if G is unimodular then the conditions (FC) and (A) may be written in the two-sided form as a straightforward analysis of [15] shows.

We now use the (FC_{loc}) property to obtain the existence of a sequence of subsets of G which behave in a similar way to the sets $\{0,1,\ldots n\}$ in the additive semigroup of non-negative integers.

Theorem 3.3. (c.f. [15], p.383). Let G be a σ-compact amenable group. Then there exists a sequence $\{U_n\}$ of compact sets satisfying

(i) $0 < |U_n| < \infty$

(ii) $U_n \subset U_{n+1}$

(iii) $G = \bigcup_n U_n$

(iv) For every non-empty compact set $K \subset G$,
$\lim |U_n|^{-1} |gU_n \Delta U_n| = 0$ uniformly on K.

Proof. Since G is σ-compact we may choose an increasing sequence $\{K_n\}_{n=1}^{\infty}$ of compact neighbourhoods of the identity whose union is G and such that for all n, $K_n \subset \text{int.}(K_{n+1})$. We construct $\{U_n\}$ inductively using the (FC_{loc}) property of G (theorem 3.1.).

Choose U_1 to satisfy (FC_{loc}) for $(1, K_1)$ with $K_1 \subset U_1$. For $n > 1$, choose U_n to satisfy (FC_{loc}) for $(\frac{1}{n}, K_n)$ with $K_n \cup U_{n-1} \subset U_n$. Clearly $\{U_n\}$ satisfies (i)-(iii). If K is any non-empty compact set, $K \subset K_m$ for some m and hence $K \subset K_n$ for all $n \geqslant m$. Then for $n \geqslant m$ and $g \in K$,

$$|gU_n \Delta U_n| \leqslant \frac{1}{n} |U_n| \text{ and (iv) follows.}$$

<u>Remarks</u> (i) If G is <u>unimodular</u> the condition (iv) may be replaced by

(iv') For every non-empty compact set $K \subset G$,
$\lim |U_n|^{-1}|gU_n \Delta U_n| = \lim |U_n|^{-1}|U_n g \Delta U_n| = 0$ uniformly on K.

(ii) The same argument is valid for an amenable semigroup and we obtain the existence of a sequence $\{U_n\}$ of finite sets satisfying (i), (ii), (iii) and (iv') of the above theorem and remark.

<u>Definition 3.4</u>. If G is a σ-compact amenable group (resp. <u>unimodular</u> group or semigroup), then a sequence of compact sets $\{U_n\}$ satisfying (i)-(iv) (resp. (i)-(iii), (iv')) of theorem 3.3 and remark, will be called a <u>summing sequence</u> for G. It will be shown that the concept of a summing sequence captures precisely the significant algebraic structure of the sequence $\{\sigma_n\}$ in the additive semigroup of non-negative integers where $\sigma_n = \{0,1,\ldots,n\}$.

Theorem 3.3 may be extended somewhat. For example, Namioka ([33] p.28) has shown that for discrete groups, the sets $\{U_n\}$ may be assumed symmetric. His proof readily extends to unimodular groups. Also as shown in [15] p.383, it is possible to choose $\{U_n\}$ to satisfy (i)-(iii) and

(iv") for every non-empty compact set $K \subset G$,

$$\lim |U_n|^{-1} |KU_n \Delta U_n| = 0$$

rather than (iv) or (iv').

As will be seen when we consider the application of amenability to ergodic theory, this application may be couched in terms of summing sequences. For this reason the structure of summing sequences is important. An outstanding unsolved problem is the following: if G is generated by a compact set U containing e (so that G is σ-compact and $G = \bigcup_{n=1}^{\infty} U^n$), when is it true that $\{U^n\}$ is a summing sequence? For G connected and separable this has been solved by Kawada [26] (see also [16]) who showed that every such sequence $\{U^n\}$ is a summing sequence. This proof depends largely on the fact that the structure of connected amenable groups is essentially known ([36] p.185). However for discrete groups or semigroups, very little is known. The following simple result due to Emerson and Greenleaf is of some interest. Call a sequence $\{U_n\}$ of compact sets satisfying (i)-(iii) and (iv") a <u>strong</u> summing sequence. Then

<u>Theorem 3.5</u>. ([16] p.176-7). Let G be an amenable group generated by a compact neighbourhood U of e. Then $\{U^n\}$ is a strong summing sequence $\Leftrightarrow \lim |U^n|^{-1} |U^{n+1}| = 1$.

Proof. If $\{U^n\}$ is a strong summing sequence, then by (iv"), $\lim |U^n|^{-1}|KU^n \Delta U^n| = 0$ for all compact $K \neq \phi$. In particular if $K = U$, $|KU^n \Delta U^n| = |U^{n+1} \backslash U^n|$ so that $\lim |U^n|^{-1} [|U^{n+1}| - |U^n|] = 0$

i.e. $\lim |U^n|^{-1}|U^{n+1}| = 1$

Conversely if $\lim |U^n|^{-1}|U^{n+1}| = 1$ then clearly for all $k \geqslant 1$, $\lim |U^n|^{-1}|U^{n+k}| = 1$.

Choose K compact, non-empty. Then for k sufficiently large, $K \subset U^k$. Now we have

$$\begin{aligned} |U^n \cap KU^n| + |U^n \backslash KU^n| &= |U^n| \\ &\leqslant |KU^n| \\ &= |KU^n \backslash U^n| + |KU^n \cap U^n| \end{aligned}$$

which implies that $|U^n \backslash KU^n| \leqslant |KU^n \backslash U^n|$.

$$\begin{aligned} \text{Hence } |KU^n \Delta U^n| &= |KU^n \backslash U^n| + |U^n \backslash KU^n| \\ &\leqslant 2|KU^n \backslash U^n| \\ &\leqslant 2|U^{n+k} \backslash U^n| \end{aligned}$$

$$\begin{aligned} \text{so that } \lim |U^n|^{-1}|KU^n \Delta U^n| &\leqslant \lim 2|U^n|^{-1} [|U^{n+k}| - |U^n|] \\ &= 0 \end{aligned}$$

and $\{U^n\}$ is a strong summing sequence.

The following example will be needed later.

<u>Proposition 3.6</u>. Let G be the free Abelian semigroup generated by $\{g_1, \ldots, g_r\}$. Let $U_0 = \{e\}$ and for $n \geqslant 1$

define $U_n = \{g_1^{\alpha_1} \ldots g_r^{\alpha_r}: \max\limits_{1 \leqslant i \leqslant r} \alpha_i \leqslant n\}$.

Then $\{U_n\}$ is a summing sequence.

Proof. It is obvious that properties (i)-(iii) of a summing sequence hold. We verify (iv').

Let $g \in G$ and suppose $g = g_1^{\beta_1} \ldots g_r^{\beta_r}$.
Then for $n \geqslant 1$, $gU_n = \{g_1^{\alpha_1+\beta_1} \ldots g_r^{\alpha_r+\beta_r}: 0 \leqslant \alpha_i \leqslant n\}$
so that

$$gU_n \cap U_n = \{g_1^{\alpha_1+\beta_1} \ldots g_r^{\alpha_r+\beta_r}: 0 \leqslant \alpha_i \leqslant n - \beta_i\}$$

whenever n is sufficiently large i.e. $n \geqslant \max \beta_i$

Now it is clear that $|U_n| = (n+1)^r$ and by the same argument, $|gU_n \cap U_n| = \prod\limits_{n=1}^{r} (n - \beta_i + 1)$.
Hence

$$|U_n|^{-1} |gU_n \cap U_n| = \prod_{i=1}^{r} n^{-1}(n - \beta_i + 1) \text{ so that}$$

$$\lim |U_n|^{-1} |gU_n \cap U_n| = 1.$$

By commutativity, $\lim |U_n|^{-1} |U_n g \cap U_n| = 1$ and these last two statements are equivalent to (iv').

It should also be noted in this case that if we put $V_0 = \{e\}$, $V = \{g_1, \ldots, g_r\}$, then $\{V^n\}$ is a summing sequence. We shall not need this result and we omit the (simple) proof. This should be compared, however, with a similar result in [21] (Theorem 3.6.6).

It is easy to obtain more examples of this kind. For example if G is the additive semigroup of non-negative integers and $\{a_n\}$ any increasing sequence of non-negative integers such that $\lim a_n = \infty$ then $\{U_n\} = \{0,1,\ldots,a_n\}$ defines a summing sequence. This idea may be used to give various generalizations of the previous proposition. If we consider a compact (semi)-group G then it has a trivial summing sequence $\{U_n\}$ where $U_n \equiv G$. If now G is taken to be a σ-compact, locally finite group then G is expressible as $G = \bigcup_{n=1}^{\infty} G_n$ where the G_n's are an increasing sequence of compact subgroups. It is easy to verify that in this case the G_n's define a summing sequence for G. Other examples abound.

§4 ALMOST CONVERGENCE

The idea of almost convergence was introduced by Lorentz [29] for bounded sequences of numbers. It was extended to amenable semigroups by Day [4] where various alternative characterizations were proved. Later, Dye [13] used a somewhat different definition to obtain results on the ergodic mixing theorem. Dye's definition was subsequently used by Douglass [8] (see especially theorem 4.1) to obtain a generalization to amenable semigroups of Lorentz' result which characterizes almost convergence as a uniform limit of averages. In this section we look in some detail at Douglass' theorem. We obtain an alternative proof and various generalizations of this theorem, embedding it in a more general context and showing incidentally some distinctions which are lacking in the original proof.

Definition 4.1. A function $f \in L_\infty(G)$ (G an amenable group or semigroup) is left almost convergent (l.a.c.) to s if for every left invariant mean m on $L_\infty(G)$ we have $m(f) = s$. Similarly we may define right almost convergence (r.a.c.).

Definition 4.2. $f \in L_\infty(G)$ is almost convergent (a.c.) if it is both l.a.c. and r.a.c. (to the same scalar s since two-sided invariant means exist by theorem 1.6 and remark).

This last definition is the one employed by Dye and Douglass. Day's original definition is slightly different.

Definition 4.3. f is (Day) almost convergent (D.a.c.) to s if for every two-sided invariant mean m we have $m(f) = s$.

It is clear that (a.c.) $\Rightarrow$ (D.a.c.) we shall later find conditions under which the converse is true.

In [1], Arens showed how to define an associative multiplication in the second conjugate space B^{**} of a

Banach algebra B. This multiplication (the Arens product) makes B^{**} a Banach algebra and is an extension of the multiplication in B. For our purposes we put $B = L_1(G)$ so that $B^{**} = L_\infty(G)^*$. Multiplication is introduced into $L_\infty(G)^*$ as follows

For $m \in L_\infty(G)^*$, $f \in L_\infty(G)$, define $m \circ f \in L_\infty(G)$ by

$$\langle m \circ f, x \rangle = m(x^* * f) \quad \text{for all } x \in L_1(G).$$

For $m, n \in L_\infty(G)^*$, define $m \circ n \in L_\infty(G)^*$ by

$$\langle m \circ n, f \rangle = \langle m, n \circ f \rangle \quad \text{for all } f \in L_\infty(G).$$

The Arens product yields important information on the behaviour of invariant means. The following result is typical

Proposition 4.4. Let $m, n \in L_\infty(G)^*$. Then

(i) $m \circ n$ is left invariant if m is left invariant. It is right invariant if n is right invariant.

(ii) If n is left (right) invariant and m is a mean then $m \circ n = n$ ($n \circ m = n$).

Remark. To say, for example, that m is left invariant is to require that $m(\alpha * f) = m(f)$ for all $\alpha \in F$, $f \in L_\infty(G)$; m need not be a mean. Similarly for right invariance.

We omit the proof of this proposition since it depends on some fairly detailed analysis of the Arens product (see e.g. [39], lemma 4.1). This result now yields a useful characterization of almost convergent functions. Let S denote the subspace of all functions in $L_\infty(G)$ which can be represented in the form $\sum_{i=1}^{n} (f_i - \mu_i * f_i)$ where $f_i \in L_\infty(G)$, $\mu_i \in M_1^+(G)$, $1 \leq i \leq n$. Denote by $\bar{S}$ the L_∞-norm closure of S. Then we have

<u>Theorem 4.5</u>. Let C denote the constants in $L_\infty(G)$. Then $C \oplus \bar{S}$ is the space of all left almost convergent functions in $L_\infty(G)$ with f l.a.c. to s iff $f \in s\,1 + \bar{S}$.

<u>Remarks</u> (i) This result is essentially due to Wong ([39], theorem 7.3) where however S is taken to be those functions of the form $\sum_{i=1}^{n} f_i - \alpha_i * f_i$ where $f_i \in L_\infty(G)$, $\alpha_i \in F$. Our definition will be somewhat more useful in obtaining the Lorentz characterization.

(ii) If we denote by T the subspace of all functions in $L_\infty(G)$ representable in the form $\sum_{i=1}^{n} f_i - f_i * \tilde{\mu}_i$ with $f_i \in L_\infty(G)$, $\mu_i \in M_1^+(G)$, then a similar result may be obtained for right almost convergence.

Proof of Theorem 4.5. Suppose firstly that f is l.a.c. to zero. We show that $f \in \bar{S}$. If not, the Hahn-Banach theorem implies the existence of some $m \in L_\infty(G)^*$ such that

$m(\bar{S}) = 0$ and $m(f) \neq 0$. We may assume that $m = \gamma_1 m_1 - \gamma_2 m_2$ where m_1, m_2 are means of $L_\infty(G)$; γ_1, γ_2 are scalars and note that m is left invariant. If now n is any left invariant mean, then by proposition 4.4,

$$m = n \circ m = \gamma_1 (n \circ m_1) - \gamma_2 (n \circ m_2)$$

and both $n \circ m_1$ and $n \circ m_2$ are left invariant means. Hence $m(f) = 0$ which is a contradiction so that $f \in \bar{S}$. Conversely if m is any left invariant mean, $m(\bar{S}) = 0$. Hence $\bar{S} = \{f \in L_\infty(G) : f \text{ is l.a.c. to } 0\}$.

Now f is l.a.c. to s iff $f - s1$ is l.a.c. to 0, i.e. iff $f \in s1 + \bar{S}$. To show that the sum $C + \bar{S}$ is direct, let $s1 \in \bar{S}$. Then if m is any left invariant mean, $s = m(s1) = 0$ which proves the result.

This characterization of almost convergence may now be applied to give a result which may be regarded as a generalization of Douglass' theorem ([8], theorem 4.1).

<u>Theorem 4.6</u>. Let $f \in L_\infty(G)$. Then a necessary and sufficient condition that f be l.a.c. (r.a.c.) to s is that for all nets $\{\alpha_\gamma\}$ of finite means, norm-convergent to left (right) invariance we have

$$\lim \int_G \alpha_\gamma(g)(f * \tilde{\mu})(g)dg = s \text{ uniformly on } M_1^+(G) \quad (*)$$

$$(\lim \int_G \alpha_\gamma(g)(\mu * f)(g)dg = s \text{ uniformly on } M_1^+(G)).$$

Proof. We consider the left case only. Suppose firstly that f is the form $f = f_1 - \nu * f_1$ with $f_1 \in L_\infty(G)$, $\nu \in M_1^+(G)$. Then if $\mu \in M_1^+(G)$,

$$\begin{aligned}
\int_G \alpha_\gamma(g)(f*\tilde{\mu})(g)dg &= \int_G (\alpha_\gamma*\mu)(g)f(g)dg \\
&= \int_G (\alpha_\gamma*\mu)(g)f_1(g)dg \\
&\quad - \int_G (\alpha_\gamma*\mu)(g)(\nu*f_1)(g)dg \\
&= \int_G [\alpha_\gamma*\mu - \nu*\alpha_\gamma*\mu](g)f_1(g)dg.
\end{aligned}$$

$$\begin{aligned}
\therefore \left|\int_G \alpha_\gamma(g)(f*\tilde{\mu})(g)dg\right| &\leqslant ||(\alpha_\gamma - \nu*\alpha_\gamma)*\mu||\ ||f_1|| \\
&\leqslant ||\alpha_\gamma - \nu*\alpha_\gamma||\ ||f_1||
\end{aligned}$$

and by lemma 2.3, $\lim ||\alpha_\gamma - \nu*\alpha_\gamma|| = 0$ so that the result follows for such a function f. Clearly the result also follows for all $f \in S$. If now $f \in \bar{S}$ (i.e. f is l.a.c. to 0) then for $\varepsilon > 0$, choose $f_0 \in S$ such that $||f - f_0|| < \varepsilon$. Then

$$\begin{aligned}
\left|\int_G \alpha_\gamma(g)(f*\tilde{\mu})(g)dg\right| &\leqslant \left|\int_G \alpha_\gamma(g)(f_0*\tilde{\mu})(g)dg\right| \\
&\quad + \left|\int_G \alpha_\gamma(g)([f - f_0]*\tilde{\mu})(g)dg\right| \\
&\leqslant \left|\int_G \alpha_\gamma(g)(f_0*\tilde{\mu})(g)dg\right| + \varepsilon
\end{aligned}$$

and the result again follows. Finally if f is l.a.c. to s then $f - s.1 \in \bar{S}$ so that using the fact that $\int_G \alpha_\gamma(g)(1*\tilde{\mu})(g)dg = 1$ we prove necessity.

Conversely let $f \in L_\infty(G)$ satisfy condition (*). Considering if necessary $f - s.1$, we may assume from the start that $s = 0$. Now letting $\mu = \delta_h$ we find that for all nets $\{\alpha_\gamma\}$ norm-convergent to left invariance,

$$\lim \int_G \alpha_\gamma(g)f(gh)dg = 0 \text{ uniformly on } G.$$

But $\int_G \alpha_\gamma(g)f(gh)dg = \int_G \Delta(g^{-1})\alpha_\gamma(g^{-1})f(g^{-1}h)dh$

$$= (\alpha_\gamma{}^* * f)(h)$$

i.e. $\lim ||\alpha_\gamma{}^* * f|| = 0.$

Now if m is any left invariant mean,

$$|m(f)| = |m(\alpha_\gamma{}^* * f)| \leqslant ||\alpha_\gamma{}^* * f||$$

so that $m(f) = 0$ and f is l.a.c. to 0.

The proof of this theorem easily shows that the following apparently weaker result is also true.

Theorem 4.7. With the notation of theorem 4.6, f is l.a.c. (r.a.c.) to s iff

$$\lim \int_G \alpha_\gamma(g)f(gh)dg = s \text{ uniformly on } G$$

$$(\lim \int_G \alpha_\gamma(g)f(hg)dg = s \text{ uniformly on } G).$$

Looking now at the role played by the net $\{\alpha_\gamma\}$ in the proof of theorem 4.6 we obtain another apparently weaker form of the last two theorems. For brevity we consider only the last result.

Theorem 4.8. f is l.a.c. (r.a.c.) to s iff for some net $\{\alpha_\gamma\}$ norm-convergent to left (right) invariance we have

$$\lim \int_G \alpha_\gamma(g)f(gh)dg = s \text{ uniformly on } G$$

$$(\lim \int_G \alpha_\gamma(g)f(hg)dg = s \text{ uniformly on } G).$$

This last result gives one of the most manageable characterizations for almost convergence. This is because in a large number of cases such nets $\{\alpha_\gamma\}$ may be easily constructed and have a particularly simple form. The most important case is when G is σ-compact. The following lemma is basic.

Lemma 4.9. Let G be σ-compact (or a semigroup) let $\{U_n\}$ be a summing sequence for G (see definition 3.4). Denote by χ_{U_n} the characteristic function of U_n and let $\lambda_n = |U_n|^{-1}\chi_{U_n}$ be the associated weight function. Then $\{\lambda_n\}$ is norm-convergent to left invariance. If G is unimodular (or a semigroup) and recalling the two-sided definition for summing sequences in this case, then $\{\lambda_n\}$

is also norm-convergent to right invariance.

Proof. For semigroups, this is in [8], lemma 3.1. We therefore consider the group case only, noting in passing that the semigroup case is entirely similar.

Let α be a weight on G. Then for $g \in G$,

$$(\alpha * \lambda_n)(g) - \lambda_n(g) = \int_G \alpha(h)\lambda_n(h^{-1}g)dh - \lambda_n(g)$$

$$= \int_G \alpha(h)[\lambda_n(h^{-1}g) - \lambda_n(g)]dh$$

Hence $$||\alpha*\lambda_n - \lambda_n||_1 = \int_G \left|\int_G \alpha(h)[\lambda_n(h^{-1}g) - \lambda_n(g)]dh\right|dg$$

$$\leqslant \int_G\int_G \alpha(h)|\lambda_n(h^{-1}g) - \lambda_n(g)|dhdg$$

$$= |U_n|^{-1}\int_G\int_G \alpha(h)|\chi_{hU_n}(g) - \chi_{U_n}(g)|dgdh$$

$$= |U_n|^{-1}\int_G \alpha(h)[\int_G \chi_{hU_n \Delta U_n}(g)dg]dh$$

$$= |U_n|^{-1}\int_G \alpha(h)|hU_n \Delta U_n|dh$$

Fix $\varepsilon > 0$, choose K compact such that $\int_{G\backslash K} \alpha(h)dh < \varepsilon/2$ and choose n_0 such that $|U_n|^{-1}|hU_n \Delta U_n| < \varepsilon/2$ for all $h \in K$ whenever $n \geqslant n_0$. Then for $n \geqslant n_0$,

$$||\alpha*\lambda_n - \lambda_n||_1 \leqslant |U_n|^{-1}\int_{G\backslash K}\alpha(h)|hU_n \Delta U_n|dh$$

$$+ |U_n|^{-1}\int_K \alpha(h)|hU_n \Delta U_n|dh$$

$$< |U_n|^{-1}|U_n|\varepsilon/2 + \int_K \alpha(h)dh.\varepsilon/2$$

$$< \varepsilon.$$

Hence $\lim ||\alpha*\lambda_n - \lambda_n||_1 = 0$.

Now suppose that G is unimodular and that $\{U_n\}$ is a summing sequence for G. Then for α any weight in G,

$$(\lambda_n*\alpha)(g) - \lambda_n(g) = \int_G \lambda_n(h)\alpha(h^{-1}g)dh - \lambda_n(g)$$

$$= \int_G \alpha(h)[\lambda_n(gh^{-1}) - \lambda_n(g)]dh$$

using unimodularity. Hence

$$||\lambda_n*\alpha - \lambda_n||_1 = \int_G \left|\int_G \alpha(h)[\lambda_n(gh^{-1}) - \lambda_n(g)]dh\right|dg$$

$$\leqslant |U_n|^{-1}\int_G \alpha(h)|U_nh \Delta U_n|dh$$

and the proof now follows as above.

Theorem 4.8 now yields at once the following result

<u>Theorem 4.10</u>. Let G be σ-compact, unimodular and let $\{U_n\}$ be a summing sequence for G. Then $f \in L_\infty(G)$ is l.a.c. (r.a.c.) to s iff

$$\lim |U_n|^{-1}\int_{U_n} f(gh)dg = s \text{ uniformly on } G$$

$$(\lim |U_n|^{-1}\int_{U_n} f(hg)dg = s \text{ uniformly on } G).$$

We now consider Day's original idea of almost convergence. Recalling definition 4.3, $f \in L_\infty(g)$ is (Day)-almost convergent (D.a.c.) to s if $m(f) = s$ for every two-sided invariant mean m.

<u>Theorem 4.11</u>. Almost convergence coincides with Day-almost convergence iff every left invariant mean is also right invariant.

Proof. Trivially a.c. always implies D.a.c. Further if every left invariant mean is also right invariant then every left or right invariant mean will be (two-sided) invariant in which case D.a.c. $\Rightarrow$ a.c.

Suppose then that D.a.c. $\Rightarrow$ a.c. Again let

$$S = \{\sum_{i=1}^{n} (f_i - \mu_i * f_i) : f_i \in L_\infty(G),\ \mu_i \in M_1^+(G)\}$$

$$T = \{\sum_{i=1}^{n} (f_i - f_i * \tilde{\nu}_i) : f_i \in L_\infty(G),\ \nu_i \in M_1^+(G)\}$$

and let

$$V = \{\sum_{i=1}^{n} (f_i - \mu_i * f_i * \tilde{\nu}_i) : f_i \in L_\infty(G),\ \mu_i,\ \nu_i \in M_1^+(G)\}.$$

Then as we saw in theorem 4.5 and remarks, $\bar{S}$ $[\bar{T}]$ is precisely the set of functions left [right] almost convergent to 0. By the same argument and since

$$f - \mu * f * \tilde{\nu} = (f - \mu * f) + (\mu * f) - (\mu * f) * \tilde{\nu}$$

it follows that $\bar{V} = \overline{S+T}$ is precisely the set of functions D.a.c. to 0. By assumption therefore, $\bar{V} = \overline{S+T} \subseteq \bar{S} \cap \bar{T}$ from which we see that $\bar{S} = \bar{T}$. Hence if m is any left invariant mean, $m(\bar{S}) = 0$ i.e. $m(\bar{T}) = 0$ i.e. m is right invariant.

It is obvious that for compact or Abelian groups, left and right invariance coincide. It would be interesting to know for which class of groups this is true. Nothing in this direction seems to be known.

§5 DAY'S FIXED POINT THEOREM

In this section we shall prove a result which is crucial in the application of amenability to ergodic theory. This is Day's celebrated fixed point theorem, also known as the Markov-Kakutani theorem in the Abelian case. Day [5] proved this result for discrete amenable semigroups acting affinely on a class of topological vector spaces. Since then it has been extended to locally compact groups with continuous action (see e.g. [21], theorem 3.5.5). We shall prove a version of this theorem which is in a sense weaker (with linear rather than affine transformations) and in a sense stronger (measurability rather than continuity of action). The proof however is very similar to Day's.

We shall need integration of vector-valued functions in the following sense.

<u>Theorem 5.1</u>. Let G be a locally compact group (space) with left Haar measure dg, B a Banach space. Suppose that $x\colon G \to B$ is weakly measurable and that $x^*[x(g)]$ is integrable for each $x^* \in B^*$. Then there exists an $x^{**} \in B^{**}$ such that

$$x^{**}(x^*) = \int_G x^*[x(g)]\,dg \quad \text{for all} \quad x^* \in B^*.$$

We write $x^{**} = \int_G x(g)dg$.

Here weakly measurable means that $x^*[x(g)]$ is a measurable function on G for all $x^* \in B^*$. For a proof of this theorem, see [24], theorem 3.7.1. Note that if $x^{**} \in B$ rather than B^{**}, we have the more familiar Pettis integral. The following easily verifiable properties of the integral will be required

<u>Lemma 5.2</u>. Suppose that $x\colon G \to B$ is weakly measurable. Then

(i) $\int_G f(g)x(g)dg$ exists for all $f \in L_1(G)$, whenever x is (essentially) bounded.

(ii) $||\int_G x(g)dg|| \leqslant \int_G ||x(g)||dg$ if x is integrable

(iii) $\left[\int_G f(g)dg\right] y = \int_G [f(g)y]dg$ if $f \in L_1(G)$, $y \in B$

(iv) If ψ is measurable on $G \times G$ then

$$\int_G \left[\int_G \psi(h,g)dh\; x(g)dg = \int_G \left[\int_G \psi(h,g)x(g)dg\right] dh\right.$$

in the sense that if either integral exists then so does the other and the integrals are equal.

(v) If T is a bounded linear operator on B then

$$\int_G T[x(g)]\,dg = T^{**}\left[\int_G x(g)dg\right].$$

Theorem 5.3. (Fixed-Point Theorem). Let B be a Banach space, K a closed, convex, weakly compact subset of B. Suppose that G is a locally compact amenable group (or semigroup) and that $g \to U_g$ is a bounded, weakly measurable representation of G as linear operators on B such that $U_g : K \to K$ for all $g \in G$. Then K has a fixed point i.e. there exists some $x_0 \in K$ such that $U_g x_0 = x_0$ for all $g \in G$.

Remark. We consider the group case only, the proof for semigroups being virtually identical. Note also that it suffices to consider real Banach spaces so that (temporarily) we take $L_\infty(G) = L_\infty^r(G)$, the real Banach space of essentially bounded, measurable real-valued functions on G. We may also assume that $||U_g|| \leqslant 1$ for all $g \in G$.

Fix $y \in K$. Define T on B^* by $(Tx^*)(g) = x^*(U_g y)$ for all $x^* \in B^*$, $g \in G$. Since $g \to U_g$ is weakly measurable, Tx^* is a measurable function on G. Also

$|(Tx^*)(g)| = |x^*(U_g y)| \leqslant ||x^*|| \, ||y||$ so that $T: B^* \to L_\infty(G)$. Clearly T is bounded and linear.

Let Π be the canonical map from B into B^{**}. Let $K' = \Pi(K)$. K' is then a closed, convex, w^*-compact subset of B^{**}. The proof of the theorem now rests on the following lemmas.

Lemma 5.4. If m is a mean on $L_\infty(G)$, then $T^*m \in K'$.

Proof. For $x^* \in B^*$ we have

$$\begin{aligned}
(T^*m)x^* &= m(Tx^*) \\
&\leqslant \text{ess. sup } \{(Tx^*)(g) : g \in G\} \\
&= \text{ess. sup } \{x^*(U_g y) : g \in G\} \\
&\leqslant \sup \{x^*(x) : x \in K\} \\
&= \sup \{(\Pi x)(x^*) : x \in K\} \\
&\leqslant \sup \{x^{**}(x^*) : x^{**} \in K'\}
\end{aligned}$$

Now consider B^{**} in its w^*-topology. This is a locally convex space and as a subset, K is w^*-closed and convex. Then a well-known extension of the Hahn-Banach theorem (see e.g. [14], corollary 2.2.4) shows that if T^*m is not in K', there exists an $x^* \in B^*$ such that $(T^*m)x^* > \sup \{x^{**}(x^*) : x^{**} \in K'\}$ and this is a contradiction.

From this we see that if m is a mean, $\Pi^{-1}(T^*m) \in K$. Denote by M the set of means and define

$$j: M \to K \text{ by } j(m) = \Pi^{-1}(T^*m).$$

<u>Lemma 5.5.</u> Let α be a weight on G, m_α the corresponding finite mean. For $h \in G$, define $\lambda_h: L_\infty(G) \to L_\infty(G)$ by $\lambda_h(f) = {}_hf$. Then for all $h \in G$, $j(\lambda_h^* m_\alpha) = U_h(jm_\alpha)$.

Proof. For $f \in L_\infty(G)$ we have $m_\alpha(f) = \int_G \alpha(g)f(g)dg$ so that

$$\begin{aligned}(\lambda_h^* m_\alpha)(f) = m_\alpha(\lambda_h f) &= \int_G \alpha(g)(\lambda_h f)(g)dg \\ &= \int_G \alpha(g)f(hg)dg \\ &= \int_G \alpha(h^{-1}g)f(g)dg\end{aligned}$$

Now let $x^* \in B^*$. Then

$$\begin{aligned}x^*(jm_\alpha) = x^*(\Pi^{-1}T^*m_\alpha) &= (T^*m_\alpha)(x^*) \\ &= m_\alpha(Tx^*) \\ &= \int_G \alpha(g)(Tx^*)(g)dg \\ &= \int_G \alpha(g)x^*(U_g y)dg.\end{aligned}$$

From this we see that $\int_G \alpha(g)U_g y dg$ is in <u>B</u> rather than in B^{**} and that $j(m_\alpha) = \int_G \alpha(g)U_g y dg$.

But then

$$j(\lambda_h{*}m_\alpha) = \int_G \alpha(h^{-1}g)U_g y\,dg$$

$$= \int_G \alpha(g)U_{hg} y\,dg$$

$$= U_h\left[\int_G \alpha(g)U_g y\,dg\right]$$

i.e. $j(\lambda_h{*}m_\alpha) = U_h(jm_\alpha)$.

Proof of theorem 5.3. Let m be a mean. By proposition 1.3 the set of finite means is w*-dense in M, the set of all means. Since the functions j, $\lambda_h{*}$, U_h are continuous in their appropriate topologies we have by lemma 5.5

$$j(\lambda_h{*}m) = U_h(jm) \quad \text{for all } h \in G.$$

Since G is amenable, we may choose m to be left invariant so that $\lambda_h{*}m = m$ for all $h \in G$. For such a mean m,

$$j(m) = U_h(j(m)) \quad \text{for all } h \in G \quad \text{and}$$

$j(m)$ is the required fixed point in K.

CHAPTER 2

THE ERGODIC THEOREMS

§6. THE MEAN ERGODIC THEOREM

In this section we obtain a general abstract form of the classical mean ergodic theorem for amenable groups and semigroups. If we realize our group or semigroup by measure-preserving transformations on a measure space, we then obtain a more concrete version. The use of summing sequences in the σ-compact case shows quite clearly how the classical mean ergodic theorem of von Neumann depends only on the fact that the additive semigroup of non-negative integers is amenable.

Throughout this section, G denotes as usual an amenable group of semigroup.

Theorem 6.1. (Abstract Mean Ergodic Theorem). Let B be a Banach space, $g \to U_g$ a bounded weakly measurable representation of G on B. For $x \in B$, let K_x denote the (norm) closure of the set $\{\int_G \alpha(g)U_g x dg: \alpha \text{ a weight on } G\}$. Suppose that K_x is weakly compact. Then there exists a fixed point $x_0 \in K_x$ such that for any net $\{\alpha_\gamma\}$ of weights norm-convergent to right invariance we have

$$(\text{norm}) \lim \int_G \alpha_\gamma(g)U_g x dg = x_0.$$

Proof. (For groups - the semigroup case is entirely similar). We may assume that $||U_g|| \leq 1$ for all $g \in G$. The proof of lemma 5.5 ensures that $\int_G \alpha(g)U_g x dg$ is indeed an element of B rather than B** so that K_x is a well-defined closed convex subset of B with $U_g : K_x \to K_x$ for all $g \in G$. Since by hypothesis K_x is weakly compact, theorem 5.3 ensures that K_x has a fixed point x_0. If $x = 0$ then $K_x = \{0\}$ and 0 is the (unique) fixed point of K_x. Clearly then $\lim \int \alpha_\gamma(g)U_g x dg = 0$ so that we may assume that $x \neq 0$.

Fix $\varepsilon > 0$. Choose a weight α such that $||x_0 - \int_G \alpha(g)U_g x dg|| < \varepsilon/2$. Since $\{\alpha_\gamma\}$ is norm-convergent to right invariance, we can then find some γ_0 such that $||\alpha_\gamma * \alpha - \alpha_\gamma||_1 < \varepsilon/(2||x||)$ for all $\gamma \geq \gamma_0$.

Now $||x_0 - \int_G \alpha_\gamma(g)U_g x dg||$

$$\leq ||x_0 - \int_G (\alpha_\gamma * \alpha)(g)U_g x dg|| + ||\int_G (\alpha_\gamma * \alpha)(g)U_g x dg - \int_G \alpha_\gamma(g)U_g x dg||$$

$$= ||x_0 - \int_G [\int_G \alpha_\gamma(h)\alpha(h^{-1}g)dh]\, U_g x dg||$$

$$+ ||\int_G [(\alpha_\gamma * \alpha)(g) - \alpha_\gamma(g)]\, U_g x dg||$$

The second term is $\leqslant \int_G |(\alpha_\gamma * \alpha)(g) - \alpha_\gamma(g)| \; ||U_g x|| dg$

$$\leqslant ||x|| \int_G |(\alpha_\gamma * \alpha)(g) - \alpha_\gamma(g)| dg$$

$$= ||x|| \; ||\alpha_\gamma * \alpha - \alpha_\gamma||_1$$

$$< \varepsilon/2 \quad \text{if} \quad \gamma \geqslant \gamma_0$$

and the first term $= ||x_0 - \int_G \alpha_\gamma(h)[\int_G \alpha(h^{-1}g) U_g x dg] dh||$

$$= ||\int_G \alpha_\gamma(h) [x_0 - \int_G \alpha(g) U_{hg} x dg] dh||$$

$$= ||\int_G \alpha_\gamma(h) U_h [x_0 - \int_G \alpha(g) U_g x dg] dh|)$$

$$< \varepsilon/2.$$

Hence $||x_0 - \int_G \alpha_\gamma(g) U_g x dg|| < \varepsilon$ if $\gamma \geqslant \gamma_0$ so that

$$\lim \int_G \alpha_\gamma(g) U_g x dg = x_0.$$

We note also that the proof implies (i) x_0 is the <u>unique</u> fixed point of K_x and (ii) $\lim \int_G \alpha_\gamma(g) U_{hg} x dg = x_0$ uniformly on G. As a corollary we have

<u>Corollary 6.2</u>. Let $g \to U_g$ be a weakly measurable, <u>isometric</u> representation of G on a Hilbert space H. Let M be the subspace $\{x \in H: U_g x = x \; \forall g \in G\}$ and let P be the projection on M. Then for any net $\{\alpha_\gamma\}$ of finite means, norm-convergent to right invariance,

(strong operator) $\lim A_\gamma = P$

where $A_\gamma = \int_G \alpha_\gamma(g)U_g dg$.

Proof. A_γ denotes as usual the operator-valued integral defined by $A_\gamma x = \int_G \alpha_\gamma(g)U_g x dg$ where the latter integral exists by theorem 5.1.

Let $x \in H$ and define K_x as in theorem 6.1. K_x is closed and convex and hence weakly closed. Since H is reflexive, a well-known result (see e.g. [10] p. 422-425) shows that K_x is weakly compact. Now applying the previous theorem, the result will follow once we show that $x_0 = Px$ where x_0 is the (unique) fixed point of K_x.

Note firstly that for all $g \in G$, $P = U_g P$. Clearly M is invariant under each U_g and if $x \in M^\perp$, $y \in M$ then by isometry of U_g we have

$(U_g x, y) = (U_g x, U_g y) = (x,y) = 0$ so that $M^\perp$ is invariant under each U_g. Hence $P = U_g P = PU_g$. But then

$$\begin{aligned} x_0 = Px_0 &= P \lim \int_G \alpha_\gamma(g)U_g x dg \\ &= \lim \int_G \alpha_\gamma(g)(PU_g)x dg \\ &= \lim \int_G \alpha_\gamma(g)Px dg \\ &= Px \end{aligned}$$

and the result follows.

If G is now assumed σ-compact and unimodular (or a semigroup) then theorem 6.1 may be phrased as follows

<u>Theorem 6.3</u>. Let $g \to U_g$ be a bounded, weakly measurable representation of G on B. Assume that for every $x \in B$, K_x is weakly compact. Then if $\{S_n\}$ is a summing sequence for G we have

$$\text{(norm)} \lim A_n x = \text{(norm)} \lim |S_n|^{-1} \int_{S_n} U_g x dg = x_0 .$$

Proof. We note that by the unimodularity assumption and by lemma 4.9 $\{\lambda_n\} = \{|S_n|^{-1} \chi_{S_n}\}$ is norm convergent to right invariance. Now apply theorem 6.1.

(This form of the mean ergodic theorem is due to Douglass [9] in the semigroup case.)

We now specialize some of the above results to obtain a concrete generalization of the classical mean ergodic theorem.

<u>Definition 6.4</u>. Let $(X, \mathcal{S}, \mu)$ be a measure space. A transformation $t: X \to X$ is called <u>measurable</u> if $t^{-1}(E) \in \mathcal{S}$ $\forall E \in \mathcal{S}$. We denote the action of t by $x \to xt$.

<u>Definition 6.5</u>. Let t be a measurable transformation on $(X, \mathcal{S}, \mu)$. t is called <u>measure preserving (m.p.)</u> if $\mu[t^{-1}(E)] = \mu(E)$ $\forall E \in \mathcal{S}$.

<u>Definition 6.6</u>. Let t be a m.p. transformation on $(X, \mathcal{S}, \mu)$ t is called <u>invertible</u> if there exists a m.p. transformation s on X such that $s \circ t = t \circ s = 1$ where 1 denotes the identity map.

To show that our results will be non-vacuous we need

<u>Proposition 6.7</u>. Every (semi)-group can be faithfully realized as a (semi)-group of invertible m.p. transformations on a measure space which may be assumed finite.

For discrete groups this is in Dye [12]. The locally compact and semigroup case are entirely similar.

Given a measure space $(X, \mathcal{S}, \mu)$ and a m.p. transformation t on X, t induces an operator U_t on the space of measurable functions by $(U_t f(x) = f(xt)$. The following result is well-known (see e.g. [22], p. 13-14).

<u>Proposition 6.8</u>. Let t be a m.p. transformation on $(X, \mathcal{S}, \mu)$. Then for $1 \leqslant p < \infty$, the operator U_t is an isometry on $L_p(X)$. U_t is invertible iff t is invertible.

Now suppose that G is realized as a (semi)-group of m.p. transformations on $(X, \mathcal{S}, \mu)$ under the maps $x \to xg$, $g \in G$. We easily see that $g \to U_g$ defines an isometric representation of G on $L_p(X)$. We make the following blanket assumption.

<u>If $f \in L_1(X)$ then $f(xg)$ is measurable in $X \times G$.</u>

It follows that for almost $x \in X$, $f(xg)$ is a measurable function on G, integrable over every compact subset of G. Further if $f \in L_p(X)$, $h \in L_q(X)$, $p \geq 1$, $p^{-1} + q^{-1} = 1$ then $\int_X f(xg)h(x)d\mu(x)$ in a measurable function of g. This shows that for $1 \leq p < \infty$, $g \to U_g$ is weakly measurable.

<u>Definition 6.9</u>. A function f on X is called <u>G-invariant</u> if for all $g \in G$, $f(x) = f(xg)$ a.e. Clearly this is equivalent to asking that $U_g f = f \ \forall g \in G$ i.e. that f is a fixed point.

With the notation introduced we now have

<u>Theorem 6.10</u>. (Concrete Mean Ergodic Theorem). For every p, $1 < p < \infty$ and for every function $f \in L_p(X)$ there exists an $f^* \in L_p(X)$ such that for all nets $\{\alpha_\gamma\}$ of finite means on G, norm-convergent to right invariance we have

$$\lim A_\gamma f = \lim \int_G \alpha_\gamma(g) U_g f dg = f^*$$

in the mean of order p. If $\mu(X) < \infty$ then this is also true for $p = 1$. The limit function f^* is G-invariant.

Proof. We apply theorem 6.1. If $1 < p < \infty$ then $L_p(X)$ is reflexive so that the closed convex set K_f is weakly compact and the result follows. Suppose now that $p = 1$

and $\mu(X) < \infty$. For $f \in L_1(X)$ it is easily seen that the functions in K_f are uniformly equi-integrable ($\forall \varepsilon > 0$, there exists $\delta > 0$ such that if $\mu(E) < \delta$ then $\int_E |h(x)| d\mu(x) < \varepsilon \quad \forall h \in K_f$). It follows from a standard result (e.g. [10] p.294) that in this case too, K_f is weakly compact and theorem 6.1 applies again.

In certain cases the limit f^* can be identified more easily

Definition 6.11. The action of G on $(X, \mathcal{S}, \mu)$ is called ergodic (G is ergodic) if the only sets $E \in \mathcal{S}$ for which $E = Eg^{-1}$ for all g are such that $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

Proposition 6.12. G is ergodic iff the only measurable G-invariant functions are the constant functions.

Proof. (Adapted from Halmos [22].) If the only G-invariant functions are constant then for no non-trivial set $E \in \mathcal{S}$ can we have $U_g \chi_E = \chi_E$ for all g i.e. $E = Eg^{-1}$ for all g. So G is ergodic.

Suppose conversely that G is ergodic and let f be an invariant measurable function. By considering real and imaginary parts if necessary we may assume that f is real valued.

For k, n integers ($n \geqslant 0$) define

$$X(k,n) = \{x: k/2^n \leqslant f(x) \leqslant (k+1)/2^n\}.$$

$X(k,n)$ is measurable and invariance of f implies invariance of $X(k,n)$. Since G is ergodic, for each n all but one of the sets $X(k,n)$ has measure zero. Let $X(k_n,n)$ denote that one set which must satisfy $\mu(X\backslash X(k_n,n)) = 0$. The result now follows by forming the intersection of all the $X(k_n,n)$'s.

<u>Corollary 6.13</u>. (i) If $\mu(X) < \infty$ then G is ergodic iff the only G-invariant functions in $L_p(X)$ $(1 \leqslant p < \infty)$ are the constant functions.

(ii) If $\mu(X) = \infty$ then G is ergodic iff the only G-invariant function in $L_p(X)$ $(1 \leqslant p < \infty)$ is the zero function.

If we assume that G is ergodic then it follows that the limit function f^* of theorem 6.10 will be 0 if $\mu(X) = \infty$ and $\frac{1}{\mu(X)} \int_X f(x)d\mu(x)$ if $\mu(X) < \infty$.

For G <u>σ-compact, unimodular</u>, the mean ergodic theorem has the following form

<u>Theorem 6.14</u>. Let $\{S_n\}$ be a summing sequence for G. Then for all $f \in L_p(X)$, $1 \leqslant p < \infty$ ($\mu(X) < \infty$ if $p = 1$) there

exists a G-invariant function $f^* \in L_p(X)$ such that

$$\lim |S_n|^{-1} \int_{S_n} f(xg)dg = f^* \text{ in the mean of order } p.$$

(For semigroups this is due to Douglass [9].) This follows immediately from theorems 6.3 and 6.10.

As a corollary to this result we can deduce the well-known ergodic theorem of Wiener [38].

<u>Corollary 6.15</u>. Let $\{\phi_1,\ldots,\phi_r\}$ be a set of commuting m.p. transformations on $(X, \mathcal{S}, \mu)$. Then

$$\lim \frac{1}{n^r} \sum_{1\leq j_1,\ldots,j_r\leq n} f(x\phi_1^{j_1}\ldots\phi_r^{j_r})$$

exists in the norm of order p.

Proof. This follows from the above theorem together with proposition 3.6.

Finally we prove a stability theorem which in the classical case and for almost everywhere convergence is due to Maharam [30]. It is instructive to see how easy this result is to verify when embedded in its natural setting.

<u>Definition 6.16</u>. Let G be σ-compact, $\{S_n\}$ a summing sequence for G. A measurable set $A \subset G$ has <u>zero density</u> if

$$\lim |S_n|^{-1}|S_n \cap A| = 0.$$

With the notation of theorem 6.14 we have for $p = 1$,

Theorem 6.17. For any set A of zero density,

$$\lim |S_n|^{-1} \int_{S_n \setminus A} f(xg)dg = f^*.$$

Proof. Clearly it suffices to show that

$$\lim |S_n|^{-1} \int_{S_n \cap A} f(xg)dg = 0.$$

But $|S_n|^{-1} || \int_{S_n \cap A} f(xg)dg||_1 \leqslant |S_n|^{-1} \int_{S_n \cap A} ||f(xg)||_1 dg$

$$\leqslant ||f||_1 |S_n|^{-1} |S_n \cap A|$$

and the result follows.

§7 THE INDIVIDUAL ERGODIC THEOREM.

In this section we show that in general, the mean convergence of the ergodic averages may be replaced by almost everywhere convergence. This gives a generalization of Birkhoff's Individual Ergodic Theorem. Since the proof of our theorem depends ultimately on a theorem of Banach which itself invokes a category argument, we restrict ourselves to σ-compact groups or semigroups. Briefly then we have the following case.

G is either a (countable, cancellative) amenable semigroup or a σ-compact, unimodular group. Suppose that G is represented by m.p. transformations on a measure space $(X, \mathcal{S}, \mu)$ (these transformations are not assumed invertible in the semigroup case). We suppose that the group action satisfies the blanket measurability requirements of §6. Let $\{S_n\}$ be a fixed summing sequence for G with normalized characteristic functions λ_n. Denoting by A_n the ergodic average operator on $L_p(X)$ defined by

$$(A_n f)(x) = |S_n|^{-1}\int_{S_n} f(xg)dg$$

we want to show that $A_n f \to f^*$ <u>almost everywhere</u>.

The central idea (and main difficulty) of the proof depends on obtaining a general form of the maximal ergodic theorem. To do this we need two intermediate lemmas. For simplicity we treat the group case only.

Suppose that η is a locally integrable function on G, i.e. η is measurable and integrable over compact subsets of G. For $r \geqslant 1$, define the operator T_r by

$$(T_r\eta)(g) = \int_G \lambda_r(h)\eta(gh)dh = |S_r|^{-1}\int_{S_r}\eta(gh)dg$$

and the operator B_r by

$$(B_r\eta)(g) = \sup_{k\leq r} |(T_k\eta)(g)|$$

let also $$(B\eta)(g) = \sup_{k} |(T_k\eta)(g)|.$$

<u>Lemma 7.1</u>. Let r be ≥ 1, η locally integrable. Then $B_r\eta$ is locally integrable and

$$\int_K [(B_r\eta)(g)]^p dg \leq \int_K |\eta(g)|^p dg$$

for all p, $1 \leq p < \infty$ and for all compact sets K.

Proof. It is straightforward to show that $B_r\eta$ is locally integrable and that $\int_K [(B_r\eta)(g)]^p dg$ exists.

Now $$(B_r\eta)(g) = \sup_{k\leq r} |(T_k\eta)(g)|$$

$$= \sup_{k\leq r} \int_G \lambda_k(h)\eta(gh)dh|$$

Let $E_1 = \{g\in K: (B_r\eta)(g) = |\int_G \lambda_1(h)\eta(gh)dh|\}$

and for $2 \leq i \leq r$ let

$$E_i = \{g \in K \setminus \bigcup_{j=1}^{i-1} E_j : (B_r\eta)(g) = |\int_G \lambda_i(h)\eta(gh)dh|\}$$

$\{E_i\}$ is a disjoint family of measurable sets with $\bigcup_{i=1}^{r} E_i = K$. Hence

$$\int_K [(B_r\eta)(g)]^p dg = \sum_{i=1}^{r} \int_{E_i} [(B_r\eta)(g)]^p dg$$

$$= \sum_{i=1}^{r} \int_{E_i} \left|\int_G \lambda_i(h)\eta(gh)dh\right|^p dg$$

$$\leqslant \sum_{i=1}^{r} \left[\int_{E_i} \lambda_i(g)dg\right] \left[\int_{E_i} |\eta(g)|^p dg\right]$$

(This last inequality follows (for groups) from the fact that $\int_G \lambda_i(h)\eta(gh)dh = \eta * \tilde{\lambda}_i(g)$ and an application of Holder's inequality yields the result. For semigroups the reasoning is similar.)

Hence

$$\int_K [(B_r\eta)(g)]^p dg \leqslant \sum_{i=1}^{r} |S_i|^{-1} |S_i \cap E_i| \int_{E_i} |\eta(g)|^p dg$$

$$\leqslant \sum_{i=1}^{r} \int_{E_i} |\eta(g)|^p dg$$

$$= \int_K |\eta(g)|^p dg.$$

which proves the lemma.

Now fix p, $1 \leqslant p < \infty$ and $f \in L_p(X)$. Define $f^0(g,x) = f(xg)$. Then f^0 is measurable on $X \times G$ and for almost all x, f^0 is locally integrable on G. Hence we may define

$$G_r(g,x) = B_r[f^0(g,x)] \text{ for } r \geqslant 1 \text{ and}$$

$$G(g,x) = B[f^0(g,x)].$$

Then we have

$$\begin{aligned} G_r(g,x) &= \sup_{k\leqslant r} |T_k f^0(g,x)| \\ &= \sup_{k\leqslant r} |T_k f^0(xg)| \\ &= \sup_{k\leqslant r} |S_k|^{-1} \left|\int_{S_k} f(xgh)dh\right| \end{aligned}$$

so that in particular

$$\begin{aligned} G_r(e,x) &= \sup_{k\leqslant r} |S_k|^{-1} \left|\int_{S_k} f(xh)dh\right| \\ &= \sup_{k\leqslant r} |A_k f(x)| \end{aligned}$$

and $\quad G(e,x) = \sup_{k} |A_k f(x)|.$

For W a neighbourhood of e, define

$$\begin{aligned} f^0_W(g,x) &= f^0(g,x) \quad \text{if} \quad g\in W \\ &= 0 \qquad\qquad \text{if} \quad g\notin W \end{aligned}$$

and for $r\geqslant 1$, let $G_{r,W}(g,x) = B_r[f^0_W(g,x)]$.
Fix such a W, $r\geqslant 1$.

<u>Lemma 7.2</u>. There exists a neighbourhood U of e such that

$G_r(g,x)\leqslant G_{r,U}(g,x)$ for all $g\in W$ and for almost all x.

Proof. We note firstly that from its definition, B_r is easily shown to be subadditive. Hence for any neighbourhood U,

$$\begin{aligned} G_r(g,x) &= B_r[f^0(g,x)] \\ &= B_r\{f_U^0(g,x) + [f^0(g,x) - f_U^0(g,x)]\} \\ &\leqslant B_r[f_U^0(g,x)] + B_r[f^0(g,x) - f_U^0(g,x)]. \end{aligned}$$

Consider the second term. Choose a neighbourhood U of e such that $WS_r \subseteq U$ and recall that

$$f^0(g,x) - f_U^0(g,x) = 0 \text{ for all } g \in U.$$

Then

$$\begin{aligned} &B_r[f^0(g,x) - f_U^0(g,x)] \\ &= \sup_{k\leqslant r} |T_k[f^0(g,x) - f_U^0(g,x)]| \\ &= \sup_{k\leqslant r} |S_k|^{-1} \left| \int_{S_k} [f^0(xgh) - f_U^0(xgh)]\,dh \right| \end{aligned}$$

Since $WS_r \subseteq U$, $WS_k \subseteq U$ for $1 \leqslant k \leqslant r$ so that if $g \in W$ and $h \in S_k$ then $gh \in U$. But then $f^0(xgh) = f_U^0(xgh)$ and $B_r[f^0(g,x) - f_U^0(g,x)] = 0$. This shows that for all $g \in W$ and almost all x,

$$G_r(g,x) \leqslant B_r[f_U(g,x)] = G_{r,U}(g,x).$$

<u>Theorem 7.3</u> (Maximal Ergodic Theorem).

Let $E_\infty = \{x: \sup_k |(A_k f)(x)| = \infty\}$. Then $\mu(E_\infty) = 0$.

Proof. Choose $\alpha > 0$, K a compact subset of G. Define

$E_\alpha = \{x: G_r(e,x) > \alpha\}$ and for U satisfying the conditions of lemma 7.2, let

$$E'_\alpha = \{(g,x): g \in K, G_{r,U}(g,x) > \alpha\}.$$

For fixed x, define

$$E'_{\alpha,x} = \{g \in K:\ G_{r,U}(g,x) > \alpha\}.$$

Let ν denote product measure in $G \times X$. We have

$$|W|\mu(E_\alpha) \leqslant \nu(E'_\alpha) = \int_X |E'_{\alpha,x}|d\mu(x).$$

Now $E'_{\alpha,x} = \{g \in K:\ B_r[f^0_U(g,x)] > \alpha\}$ so that by lemma 7.1, we have for almost all x,

$$\begin{aligned}\int_K |f^0_U(g,x)|^p dg &\geqslant \int_K \{B_r[f^0_U(g,x)]\}^p dg \\ &\geqslant \int_{E'_{\alpha,x}} \{B_r[f^0_U(g,x)]\}^p dg \\ &\geqslant \alpha^p |E'_{\alpha,x}|\end{aligned}$$

or $$|E'_{\alpha,x}| \leqslant \alpha^{-p} \int_K |f^0_U(g,x)|^p dg.$$

Hence

$$\begin{aligned}|W|\mu(E_\alpha) &\leqslant \int_X [\alpha^{-p} \int_K |f^0_U(g,x)|^p dg]\, d\mu(x) \\ &\leqslant \alpha^{-p} \int_X [\int_U |f^0_U(g,x)|^p dg]\, d\mu(x) \\ &= \alpha^{-p} \int_U [\int_X |f(xg)|^p d\mu(x)]\, dg \\ &= \alpha^{-p} \int_U [\int_X |f(x)|^p d\mu(x)]\, dg \\ &= \alpha^{-p} |U|\ ||f||_p^p.\end{aligned}$$

This relation holds for any neighbourhood U of e such that $WS_r \subseteq U$. In particular letting $U = WS_r$ we have

$$\mu(E_\alpha) \leq \alpha^{-p}||f||_p^p \; |WS_r|/|W|$$

and this inequality is valid for all neighbourhoods W of e. If now $\{W_n\}$ is a strong summing sequence (see remarks prior to theorem 3.5) we have

$$\lim_n |W_n|^{-1}|W_nS_r| = 1$$

from which we see that

$$\mu(E_\alpha) \leq \alpha^{-p}||f||_p^p.$$

Now $E_\alpha = \{x: \sup_{k \leq r} |A_kf(x)| > \alpha\}$ letting $r \to \infty$ we obtain

$$\mu\{x: \sup_k |A_kf(x)| > \alpha\} \leq \alpha^{-p}||f||_p^p$$

and letting $\alpha \to \infty$ we obtain $\mu(E_\infty) = 0$ which proves the theorem.

Now suppose that $(X, \mathcal{S}, \mu)$ is σ-finite. Let F denote the linear space of all measurable a.e. finite functions on X equipped with the topology of convergence in measure. A classical result of Banach is the following.

Banach's Theorem ([10], p.332). If $\{A_n\}$ is a sequence of continuous linear transformations from $L_p(X)$ into F, $1 \leq p < \infty$ such that

(i) sup $A_n f$ is finite a.e. for all $f \in L_p(X)$ and

(ii) lim $A_n f$ exists a.e. for f in a dense subset of $L_p(X)$ then lim $A_n f$ exists a.e. for all $f \in L_p(X)$.

With this theorem we now prove

Theorem 7.4. (Individual Ergodic Theorem) For $1 \leq p < \infty$ and for all $f \in L_p(X)$, the limit

$$\lim (A_n f)(x) = \lim |S_n|^{-1} \int_{S_n} f(xg)dg = f^*(x)$$

exists a.e. in X.

Proof. By theorem 7.3 and Banach's theorem, it suffices to consider functions f in a dense subset of $L_p(X)$.

Firstly consider functions k of the form

$$k(x) = \int_G \alpha(g)f(xg)dg - f(x)$$

where f is a measurable, essentially bounded function with finite support and α is a weight on G. We have

$$(A_n k)(x) = |S_n|^{-1} \int_{S_n} k(xg)dg$$

$$= \int_G \lambda_n(g)[\int_G \alpha(h)f(xgh)dh]\,dg - \int_G \lambda_n(g)f(xg)dg.$$

Now the first term is

$$\int_G \lambda_n(g)[\int_G \alpha(g^{-1}h)f(xh)dh]\,dg$$

$$= \int_G f(xh)[\int_G \lambda_n(g)\alpha(g^{-1}h)dg]\,dh$$

$$= \int_G f(xh)(\lambda_n * \alpha)(h)dg$$

(with similar reasoning for semigroups). Hence

$$(A_n k)(x) = \int_G f(xh)[(\lambda_n * \alpha)(h) - \lambda_n(h)]dh$$

i.e. $|(A_n k)(x)| \leqslant ||f||_\infty ||\lambda_n * \alpha - \lambda_n||_1$ a.e.

so that $\lim (A_n k)(x) = 0$ a.e.

Now suppose that $\ell \in L_p(X)$ and that ℓ is (almost) G-invariant i.e. $\ell(xg) = \ell(x)$ for almost all x and g. Then

$$(A_n \ell)(x) = |S_n|^{-1} \int_{S_n} \ell(xg)dg = \ell(x) \quad \text{a.e.}$$

It now remains to show that the linear span of the functions k and ℓ are dense in $L_p(X)$. Since X is σ-finite we know that for $1 \leqslant p < \infty$, $L_p^*(X) = L_q(X)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Fix such a p.

Suppose that $j \in L_q(X)$ and that j is orthogonal to all the functions of the form k(x). Then

$$0 = \int_X j(x)k(x)d\mu(x)$$

$$= \int_X j(x)[\int_G \alpha(g)f(xg)dg - f(x)]d\mu(x)$$

$$= \int_G \alpha(g)\{\int_X j(x)[f(xg) - f(x)]d\mu(x)\}dg.$$

Since this is true for all weights α, we have for almost all g

$$\int_X j(x)f(xg)d\mu(x) = \int_X j(x)f(x)d\mu(x)$$

$$= \int_X j(xg)f(xg)d\mu(x).$$

i.e. $\int_X f(xg)[j(x) - j(xg)]d\mu(x) = 0$

and this is true for all bounded f with finite support Hence $j(x) = j(xg)$ for almost all x, g.

Now suppose that j is orthogonal to every ℓ as well. Since j is (almost) G-invariant so is $\phi \circ j$ for every bounded continuous ϕ. If in particular ϕ vanishes outside a neighbourhood of the origin then $\phi \circ j \in L_p(X)$ and hence j is orthogonal to such $\phi \circ j$'s. This implies that $j = o$ i.e. that the linear span of the functions k and ℓ are dense in $L_p(X)$. This proves the theorem.

§8 THE ERGODIC MIXING THEOREM.

A m.p. transformation T on a measure space $(X, \mathcal{S}, \mu)$ with $\mu(X) = 1$ is called <u>weakly mixing</u> if

(A) $\lim \frac{1}{n} \sum_{k=0}^{n-1} |\mu(E \cap T^{-k}F) - \mu(E)\mu(F)| = 0$

for all measurable sets E, F.

If T is invertible then the so-called mixing theorem of ergodic theory ([22], p.39) establishes the equivalence of (A) with each of the following two conditions

(B) The unitary operater U on $L_2(X)$ induced by T has continuous spectrum on the subspace of $L_2(X)$ orthogonal to the constant functions.

(C) The Cartesian square $T \times T$ on the product space $(X \times X, \mathcal{S} \times \mathcal{S}, \mu \times \mu)$ is ergodic.

In [13], Dye obtained an abstract mixing theorem considerably more general than the above. Instead of the semigroup of non-negative integers, he considered an arbitrary amenable topological semigroup. The operation $\frac{1}{n} \sum_{k=0}^{n-1}$ is replaced by almost convergence and the condition that U has continuous spectrum is replaced by the condition that the representation U_g has no finite-dimensional subrepresentation.

In this section we show that the condition of amenability can be entirely removed and we obtain a form of Dye's theorem valid for arbitrary topological semigroups. Furthermore the result will be proved by completely elementary methods, relying only on an ergodic theorem of Birkhoff and properties of positive definite functions. Since however we deal with semigroups rather than groups some technical difficulties (not present in the group case) arise.

By a topological semigroup, we mean a semigroup S which is also a topological space such that multiplication is separately continuous.

The following ergodic theorem is due to Birkhoff [2] and is analogous to Day's fixed point theorem without the amenability condition.

<u>Theorem 8.1</u>. Let H be a Hilbert space, S a topological semigroup and $g \to U_g$ a weakly continuous representation or anti-representation of S on H such that $||U_g|| \leq 1$ for all $g \in S$. Let $M = \{x \in H: U_g x = x \text{ for all } g \in S\}$ and for $x \in H$, let K_x be the closed convex hull of $\{U_g x: g \in S\}$. Then for all $x \in H$, $K_x \cap M$ consists of exactly one point which is simultaneously

(i) the projection of x on M

(ii) the point of K_x with minimum norm.

Proof. (Adapted from Godement [20], p.60). We consider $g \to U_g$ to be a representation, the anti-representation case being entirely similar.

Fix $x \in H$. K_x being closed and convex, there exists a unique $x_0 \in K_x$ such that

$$||x_0|| = \inf \{||y||: y \in K_x\}.$$

Since $x_0 \in K_x$ and since $U_g: K_x \to K_x$ for all $g \in S$ then by uniqueness of x_0 and the fact that $||U_g x_0|| \leqslant ||x_0||$ we have $U_g x_0 = x_0$ for all $g \in S$ i.e. $x_0 \in K_x \cap M$.

Let P be the projection on M. We want to show that $x_0 = Px$. Firstly we make some observations which will be useful later. If $y \in M$ then $U_g y = y$ for all $g \in S$ so that

$$||y||^2 = (y,y) = (U_g y,y) = (y,U_g{}^*y) \leqslant ||y||\ ||U_g{}^*y|| \leqslant ||y||^2$$

so that $(y,U_g{}^*y) = ||y||\ ||U_g{}^*y||$ and $||U_g{}^*y|| = ||y||$.

Hence
$$\begin{aligned} ||y-U_g{}^*y||^2 &= ||y||^2 - (y,U_g{}^*y) - (U_g{}^*y,y) + ||U_g{}^*y||^2 \\ &= 0 \end{aligned}$$

i.e. $$U_g{}^*y = y \text{ for all } g \in S.$$

From this we see (by interchanging the role of U_g and $U_g{}^*$) that $M = \{y: U_g{}^*y = y \text{ for all } g \in S\}$. Hence M reduces every U_g so that

$$U_g P = PU_g = P.$$

Further if $z \in M^\perp$ then $U_g z \in M^\perp$ for all $g \in S$ so that $K_z \in M^\perp$.

We now show that $x_0 = Px$.

Since $x - Px \in M^\perp$ we have $K_{x-Px} \subseteq M^\perp$ so that

$$\inf \{||\sum_{i=1}^{n} \alpha_i U_{g_i}(x - Px)||: \alpha_1,\ldots\alpha_n \geqslant 0,\ \sum_{i=1}^{n} \alpha_i = 1\} = 0$$

i.e. $\inf ||\sum_{i=1}^{n} \alpha_i U_{g_i} x - Px|| = 0$

i.e. $Px \in K_x$.

$$\text{Also } ||\Sigma\alpha_i U_{g_i} x||^2 = ||\Sigma\alpha_i U_{g_i}(x - Px) + Px||^2$$

$$= ||\Sigma\alpha_i U_{g_i}(x - Px)||^2 + ||Px||^2$$

$$\geq ||Px||^2$$

so that $||Px|| \leq ||y||$ for all $y \in K_x$

i.e. $Px = x_0$.

Finally to prove uniqueness suppose that $y \in K_x \cap M$. Then $y - x_0 \in K_{x - x_0} \cap M$ so that (since $x - x_0 \in M^\perp$),

$y - x_0 \in M^\perp \cap M$

i.e. $y = x_0$.

We now introduce the idea of positive definite functions on semigroups.

<u>Definition 8.2</u>. Let S be a topological semigroup. A continuous complex-valued function ϕ on S is called <u>positive-definite</u> if there exists a Hilbert space H, a (weakly) continuous <u>isometric</u> representation $g \to U_g$ of S on H and a vector $x \in H$ such that $\phi(g) = (U_g x, x)$ for all $g \in S$.

We write $\phi \leftrightarrow (H, U_g, x)$ and note that $\phi \in CB(S)$ the Banach algebra of continuous bounded functions on S. Denote by $\mathcal{P}$ the set of positive-definite functions on S.

For groups, this definition is equivalent to the usual definition. In particular if x is cyclic then the representation $g \to U_g$ on H is uniquely defined to within unitary equivalence. For semigroups however, this need not be the case since we only assume U_g to be isometric (rather than unitary). Hence any properties of ϕ which depend upon the particular representation will have to be shown properly defined.

Proposition 8.3. Let ϕ, $\psi \in \mathcal{P}$, $\alpha \geq 0$. Then (i) $\alpha\phi$, (ii) $\phi + \psi$, (iii) $\phi\psi$, (iv) $\bar{\phi}$ all $\in \mathcal{P}$. Further $\mathcal{P}$ contains all the non-negative constant functions.

Proof. Suppose $\phi \leftrightarrow (H, U_g, x_0)$, $\psi \leftrightarrow (K, V_g, x_0')$
(i) is trivial. To show (ii), define W_g on $H \oplus K$ by

$$W_g[x, y'] = [U_g x, V_g y'] .$$

It is simple to show that $g \to W_g$ is a weakly continuous isometric representation of S on $H \oplus K$. Also

$$\begin{aligned}(\phi + \psi)(g) = \phi(g) + \psi(g) &= (U_g x_0, x_0) + (V_g x_0', x_0') \\ &= ([U_g x_0, V_g x_0'], [x_0, x_0']) \\ &= (W_g[x_0, x_0'], [x_0, x_0'])\end{aligned}$$

and $\phi + \psi \in \mathcal{P}$.

To prove (iii), let $H \otimes K$ denote the tensor product of H and K. Then

$$(\phi\psi)(g) = (U_g \otimes V_g(x_0 \otimes x_0'), x_0 \otimes x_0') \text{ and } \phi\psi \in \mathcal{P}.$$

Consider (iv). Let J be a conjugation of H, i.e. J is a conjugate linear operator such that $J^2 = I$ and $(Jx, Jy) = (y,x)$ for all $x, y \in H$. For $g \in S$ define $U_g^J = JU_gJ$. It is immediate that $g \to U_g^J$ is a weakly continuous isometric representation of S. But then

$$\begin{aligned}\overline{\phi(g)} &= (x_0, U_g x_0)\\ &= (JU_g x_0, Jx_0)\\ &= (U_g^J Jx_0, Jx_0)\end{aligned}$$

so that $\bar{\phi}$ is positive definite.
Finally the trivial representation $U_g \equiv I$ together with (i) shows that all non-negative constant functions $\in \mathcal{P}$.

The proofs of (i) and (ii) imply that every finite linear combination of positive definite functions is of the form $(U_g x, y)$. We therefore have

<u>Theorem 8.4</u>. The set V of all finite linear combinations of positive definite functions is a left and right invariant subalgebra of $CB(S)$, closed under complex conjugation and containing the constants.

Proof. By proposition 8.3 and the remarks above we need only check left and right invariance of V. This is trivial however since if $\phi \in V$, $\phi(g) = (U_g x, y)$ and $h \in S$ then

$${}_h\phi(g) = \phi(hg) = (U_{hg}x,y) = (U_g x,\ U_h{}^*y) \in V \text{ and}$$

$$\phi_h(g) = \phi(gh) = (U_{gh}x,y) = (U_g(U_h x),y) \in V.$$

Now let $f \in CB(S)$. Denote by K_f^ℓ the (norm) closed convex hull of $\{{}_hf : h \in S\}$ and by K_f^r the (norm) closed convex hull of $\{f_h : h \in S\}$.

Definition 8.5. f is called left (right) ergodic if $K_f^\ell(K_f^r)$ contains a constant function. f is called ergodic if it is both left and right ergodic.

Lemma 8.6. If f is ergodic then K_f^ℓ and K_f^r contain precisely one and the same constant.

Proof. By assumption there exist constants $m_1 \in K_f^\ell$ and $m_2 \in K_f^r$. Fix $\varepsilon > 0$ and choose $\alpha_1,\ldots,\alpha_n$, $\beta_1,\ldots,\beta_m$ positive scalars with $\Sigma\alpha_i = \Sigma\beta_j = 1$ and $g_1,\ldots,g_n$, $h_1,\ldots,h_m \in S$ such that

$$||\sum_i \alpha_i f(g_i g) - m_1||_\infty < \varepsilon \text{ and } ||\sum_j \beta_j f(gh_j) - m_2||_\infty < \varepsilon.$$

Then

$$\begin{aligned} ||\sum_{i,j} \alpha_i\beta_j f(g_ih_j) - m_2||_\infty &= ||\sum_i \alpha_i [\Sigma\beta_j f(g_ih_j) - m_2]||_\infty \\ &\leqslant \sum_i \alpha_i ||\sum_j \beta_j f(g_ih_j) - m_2||_\infty \\ &< \varepsilon \end{aligned}$$

and similarly $||\sum_{i,j} \alpha_i\beta_j f(g_ih_j) - m_1||_\infty < \varepsilon$.

i.e. $m_1 = m_2$.

We write m(f) for this unique constant.

<u>Proposition 8.7.</u> Every positive definite function is ergodic. If $\phi \in \mathcal{P}$ with $\phi \leftrightarrow (H, U_g, x)$ then $m(\phi) = ||Px||^2$ where P is the projection onto the subspace M of invariant elements of H.

Proof. We keep the notation of theorem 8.1 and note that $U_gP = PU_g = P$ for all $g \in S$. If $\psi \in K_\phi^r$ then ψ is a (uniform) limit of functions of the form

$$\sum_i \alpha_i \phi(gh_i) \qquad \alpha_i > 0,\ \Sigma\alpha_i = 1,\ h_i \in S$$

$$= \sum_i \alpha_i (U_g U_{h_i} x, x)$$

$$= (U_g [\sum_i \alpha_i U_{h_i} x], x)$$

i.e. ψ is a limit of functions of the form $(U_g z, x)$ where $z \in K_x$. Now let $z = Px$ so that by theorem 8.1, $Px \in K_x$ and hence

$$(U_g Px, x) = (Px, x) = ||Px||^2 \in K_\phi^r.$$

We now show that $||Px||^2 \in K_\phi^\ell$ and here the semigroup case presents a slight difficulty. Consider the <u>anti</u>-representation $g \to U_g^*$ on H. As noted in theorem 8.1, $M = \{y: U_g^* y = y \text{ for all } g \in S\}$. Let L_x denote the closed convex hull of $\{U_g^* x: g \in S\}$ so that $Px \in L_x \cap M$.

Hence if $\psi \in K_\phi^\ell$, ψ is a (uniform) limit of functions of the form

$$\sum_i \alpha_i \phi(h_i g) \qquad \alpha_i > 0,\ \Sigma\alpha_i = 1,\ h_i \in S$$

$$= (\sum_i \alpha_i U_{h_i} U_g x,\ x)$$

$$= (U_g x,\ \sum_i \alpha_i U_{h_i}^* x)$$

i.e. ψ is a limit of functions of the form $(U_g x, z)$ where $z \in L_x$. By the above reasoning we can put $z = Px \in L_x$ so that

$$(U_g x,\ Px) = (PU_g x,\ x) = (Px,\ x) = ||Px||^2 \in K_\phi^\ell.$$

Hence ϕ is ergodic and $m(\phi) = ||Px||^2$.

We see also that the actual representation (H, U_g, x) defining ϕ is irrelevant since by lemma 8.6, $m(\phi)$ is unique.

Using proposition 8.7 we now prove the existence of an invariant mean on V.

<u>Theorem 8.8</u>. There exists a unique linear functional m on V satisfying

(1) $|m(\phi)| \leqslant ||\phi||_\infty$

(2) $m(1) = 1$

(3) $m({}_g\phi_h) = m(\phi)$ for all $g, h \in S$.

(4) If $\phi \in \mathcal{P}$ or $\phi \geqslant 0$ then $m(\phi) \geqslant 0$.

Proof. For $\phi \in \mathcal{P}$ we define $m(\phi)$ as in proposition 8.7 by $m(\phi) = ||Px||^2 = (Px, x)$. If $\phi, \psi \in \mathcal{P}$ then the construction

of $\phi+\psi$ in proposition 8.3 shows that $m(\phi+\psi) = m(\phi)+m(\psi)$ and similarly $m(\alpha\phi) = \alpha m(\phi)$ if $\alpha \geqslant 0$.
Now any function ψ in V can be written in the form

$\psi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4)$ so that defining $m(\psi) = m(\phi_1) - m(\phi_2) + i[m(\phi_3) - m(\phi_4)]$ it is easily seen that m is a well defined linear functional on V. Moreover if $\psi(g) = (U_g x, y) \in V$ we have

$$m(\psi) = (Px, y) \qquad \text{(A)}$$

As in proposition 8.7 we can show that if $\phi \in V$ (not necessarily $\mathcal{P}$) then $m(\phi) \in K_\phi^\ell \cap K_\phi^r$ from which we deduce that $|m(\phi)| \leqslant ||\phi||_\infty$ which proves (1). (2) is trivial. (3) follows from (A) and the fact that $PU_g = U_g P = P$. (4) follows from proposition 8.7 and (1) and (2) (any linear functional satisfying (1) and (2) will be positive).

To prove uniqueness, let m' be any linear functional on V satisfying (1)-(4). Define

$$\langle x,y \rangle = m'(U_g x,y)$$

where $g \to U_g$ is some isometric representation of S on a Hilbert space H. Then $\langle\ ,\ \rangle$ is a bounded bilinear form on H so that there exists a bounded (self adjoint) operator A on H such that $\langle x,y\rangle = (Ax,y)$.
Property (3) shows that for all $g \in S$,

$$AU_g = A = U_g A \qquad \text{(B).}$$

For $x \in H$, $\alpha_1, \dots \alpha_n > 0$, $\sum_i \alpha_i = 1$, $g_1, \dots g_n \in S$ we have

$$A(\sum_i \alpha_i U_{g_i} x) = \Sigma \alpha_i Ax = Ax$$

so that by continuity, $Ay = Ax$ for all $y \in K_x$. In particular with $y = Px$ we obtain

$$AP = A = PA.$$

Now by (B) it follows from a standard separation theorem that for all x, $Ax \in K_x$. Hence we may use the fact that $P(\sum_i \alpha_i U_{g_i} x) = Px$ to obtain

$$PA = P = AP.$$

Hence $A = P$ and m is unique which proves the theorem.

Now let $g \to U_g$ be a weakly continuous isometric representation of S on a Hilbert space H. Let m denote the (unique) invariant mean on V. Fix $x_0 \in H$ and for $x, y \in H$ define

$$\langle x,y \rangle = m[(x, U_h x_0)(U_h x_0, y)].$$

[Note that by proposition 8.3 $\phi(h) = (x, U_h x_0)(U_h x_0, y) \in V$ so that $\langle x,y \rangle$ is well defined]. $\langle \, , \, \rangle$ is a bounded bilinear form on H so that there exists a bounded linear operator A on H such that

$$\langle x,y \rangle = (Ax,y) = m[(x, U_h x_0)(U_h x_0, y)]$$

<u>Lemma 8.9</u>. A is a positive, compact operator which commutes with every U_g.

Proof. (For groups this is in Godement [20], p.61-63). Since $(Ax,x) = m[|(U_h x_0, x)|^2]$, by property (4) of theorem 8.8, A is positive. We show now that A commutes with every U_g. We have

$$\begin{aligned}(U_g Ax, y) &= (Ax, U_g^* y)\\ &= m[(x, U_h x_0)(U_h x_0, U_g^* y)]\\ &= m[(x, U_h x_0)(U_{gh} x_0, y)]\\ &= m[(U_g x, U_{gh} x_0)(U_{gh} x_0, y)]\end{aligned}$$

(since U_g is isometric)

$$= m[(U_g x, U_h x_0)(U_h x_0, y)]$$

(by (3) of theorem 8.8)

$$= (AU_g x, y)$$

so that A commutes with every U_g.

Finally we show that A is compact. Let $\{y_n\} \subseteq H$ be such that (weak) $\lim y_n = 0$. We show that $\lim ||Ay_n|| = 0$.

Let B be the positive square root of A and note that B commutes with every U_g. We have

$$\begin{aligned}||Ay_n||^2 &= (Ay_n, Ay_n)\\ &= (ABy_n, By_n)\\ &= m[|(U_h x_0, By_n)|^2]\end{aligned}$$

Let $\phi_n(h) = |(U_h x_0, By_n)|^2$ $(\in V)$

$$= (By_n, U_h x_0)(U_h x_0, By_n).$$

Let J be a conjugation on H and as usual write $U_g{}^J = JU_gJ$. Then $\phi_n(h) = (U_h{}^J Jx_0, JBy_n)(U_h x_0, By_n)$

$$= ((U_h \otimes U_h{}^J)(x_0 \otimes Jx_0), By_n \otimes JBy_n)$$

so that

$$||Ay_n||^2 = m(\phi_n)$$
$$= (Q(x_0 \otimes Jx_0), By_n \otimes JBy_n)$$

where Q is the projection onto the subspace of $H \otimes H$ of all elements invariant under the representation $g \to U_g \otimes U_g{}^J$. Since $\{y_n\}$ is weakly convergent to 0 in H, it follows that $By_n \otimes JBy_n$ is weakly convergent to 0 in $H \otimes H$. i.e. $\lim ||Ay_n||^2 = 0$ and A is compact.

We can now prove our Mixing Theorem.

Theorem 8.10. Let $g \to U_g$ be a weakly continuous isometric representation of S on H and let J be a conjugation of H. Let Q be the projection onto the subspace of $H \otimes H$ of all elements invariant under $g \to U_g \otimes U_g{}^J$. Then the following conditions are equivalent

(i) $m[|(U_g x, y)|^2] = 0$ for all $x, y \in H$.

(ii) U_g has no non-trivial finite-dimensional subrepresentation.

(iii) $Q = 0$.

Proof. (i) → (ii). Suppose that (ii) fails so that there exists a finite dimensional subspace $K \neq \{0\}$ invariant under every U_g. Then for $x, y \in K$ the function $(U_g x, y)$ and hence also $|(U_g x, y)|^2$ is almost periodic. By (i) and the fact that $m[|\phi|^2] = 0 \Rightarrow \phi = 0$ whenever ϕ is almost periodic shows that $(U_g x, y) = 0$ for all $x, y \in H$, $g \in S$. Hence $K = \{0\}$ which is a contradiction.

(ii) → (i) Fix $x_0 \in H$ and define the operator A on H by $(Ax, y) = m[(x, U_h x_0)(U_h x_0, y)]$.
By lemma 8.9, A is a positive compact operator which commutes with every U_g. Hence we may write

$A = \sum_i \lambda_i P_i$ where the P_i's are finite dimensional projections commuting with every U_g. If $M_i = P_i(H)$ are the corresponding finite-dimensional subspaces then $U_g : M_i \to M_i$ defines a finite-dimensional subrepresentation. By assumption therefore $P_i = 0$ for all i, i.e. $A = 0$. This proves (i).

(iii) → (i) If $Q = 0$ then for all $x, y \in H$,

$$\begin{aligned} m[|(U_g x, y)|^2] &= m[(U_g x, y)(U_g^J Jx, Jy)] \\ &= m[((U_g \otimes U_g^J)(x \otimes Jx), y \otimes Jy)] \\ &= (Q(x \otimes Jx), y \otimes Jy) \\ &= 0. \end{aligned}$$

(i) → (iii) Assuming (i) we have

$m[((U_g \otimes U_g^J)(x \otimes Jx), y \otimes Jy)] = 0$ for all $x, y \in H$

from which it easily follows that

$$m[((U_g \otimes U_g^J)(x \otimes u), y \otimes v)] = 0 \text{ for all } x, y, u, v \in H.$$

Hence by continuity, for all $u', v' \in H \otimes H$,

$$m[((U_g \otimes U_g^J)u', v')] = 0 = (Qu', v')$$

$$\text{i.e. } Q = 0.$$

<u>Remark</u>. In Dye's original theorem ([13] theorem 1), where S is assumed amenable, condition (i) reads "$|(U_g x, y)|$ is almost convergent to 0 for all $x, y \in H$". This is clearly equivalent to our condition (i) by the uniqueness of m and the fact that if n is a linear functional on CB(S) then for $f \in CB(S)$,

$$n(|f|^2) = 0 \Leftrightarrow n(|f|) = 0.$$

We now indicate how to obtain a more concrete form of the mixing theorem. Suppose that $(X, \mathcal{S}, \mu)$ is a finite measure space ($\mu(X) = 1$) and that S is a topological semigroup realized as a semigroup of measure-preserving transformations on X under the map $x \to xg$ in such a way as to make $g \to \mu(E \cap Fg^{-1})$ continuous for all $E, F \in \mathcal{S}$. Let U_g be the associated isometry on $L_2(X)$. Then $g \to U_g$ is a weakly continuous isometric representation of S on $L_2(X)$. Further $\phi(g) = \mu(E \cap Fg^{-1}) \in V$. Then the following conditions are equivalent

(i) $m[|\mu(E \cap Fg^{-1}) - \mu(E)\mu(F)|^2] = 0$

for all E, $F \in \mathcal{B}$.

(ii) The only finite-dimensional subrepresentation of U_g is its restriction (the identity) to the subspace of constant functions.

(iii) The semigroup of product transformations $U_g \times U_g$ on the product measure space $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu)$ is ergodic.

We omit the proof noting merely that it follows from theorem 8.10 precisely as corollary 1 followsfrom theorem 1 in [13].

CHAPTER 3

APPLICATIONS TO HARMONIC ANALYSIS

§9. INVARIANT MEANS ON GROUP VON NEUMANN ALGEBRAS

Suppose that G is a locally compact Abelian group with dual group $\hat{G}$. Under the Fourier transform map $f \to \hat{f}$, the algebra $L_1(G)$ is mapped onto a dense subset of the C^*-algebra $C_0(\hat{G})$. We say therefore that $C_0(\hat{G})$ is the enveloping C^*-algebra of $L_1(G)$. It now follows that $L_\infty(\hat{G})$ is the enveloping W^*(= von Neumann)-algebra of $L_1(G)$ or the W^*-algebra generated by $L_1(G)$. We can make this more precise as follows. $L_\infty(\hat{G})$ is a W^*-algebra on $L_2(\hat{G})$ under pointwise multiplication. The Plancherel theorem show that the Hilbert spaces $L_2(G)$ and $L_2(\hat{G})$ are isomorphic and regarding $L_\infty(\hat{G})$ as a W^*-algebra on $L_2(G)$ we see that it is simply the W^*-algebra generated by the operators L_f where $f \in L_1(G)$ and L_f is defined by

$$L_f x = f * x, \quad x \in L_2(G).$$

Moreover since $\hat{G}$ is Abelian, it is amenable so that invariant means exist on $L_\infty(\hat{G})$. We can make the following observations.

(1) If $\hat{G}$ is discrete then G is compact in which case there is a unique invariant mean on $L_\infty(\hat{G})$ and this

(being the Haar integral on $\hat{G}$) defines a faithful normal trace on $L_\infty(\hat{G})$.

(2) If G is non-discrete then $\hat{G}$ is non-compact in which case it is known (see e.g. [21], appendix 1) that there are uncountably many such means and these all vanish on $C_0(\hat{G})$.

In this section we shall show that for <u>any</u> locally compact group (not necessarily Abelian) it is possible to obtain the existence of invariant means on the group W*-algebra in such a way that the remarks (1) and (2) above carry over. The point is that although the dual group $\hat{G}$ no longer exists, nonetheless it is possible to define dual algebras which correspond to $L_1(\hat{G})$ and $L_\infty(\hat{G})$ in the Abelian case in such a way as to dualize the idea of invariant means. So although not all groups are amenable, the dual algebras of a group enjoy the property of amenability. We commence with some definitions.

Let G be a locally compact group. The completion of $L_1(G)$ in the minimal regular norm ([31], p.260) is the group C*-algebra C*(G). Since $L_1(G)$ has an approximate identity, every positive linear functional on C*(G) will be determined by a positive definite function on G. Hence denoting by B(G) the dual space of C*(G) we see that

$B(G)$ may be realized as the space of all finite linear combinations of continuous positive definite functions on G. With the dual norm and pointwise multiplication, $B(G)$ is a commutative Banach algebra - the <u>Fourier-Stieltjes algebra</u> of G. The closed ideal $A(G)$ generated by elements with compact support is the <u>Fourier algebra</u> of G. An important property of $A(G)$ is that it is precisely the set of functions of the form $x*\tilde{y}$ with $x, y \in L_2(G)$ (for all this see Eymard [17]).

Let $P = \{u \in A(G)$: u is positive definite,

$$||u|| = u(e) = 1\}.$$

For $f \in L_1(G)$, denote by L_f the bounded linear operator on $L_2(G)$ defined by $L_f x = f*x$ and denote by $W^*(G)$ the W^*-algebra on $L_2(G)$ generated by the operators L_f, $f \in L_1(G)$. Then ([17], p.210-11), $W^*(G)$ may be identified with the dual space of $A(G)$ under the map

$$\langle T, u \rangle = (Tx, y) \text{ where } u \in A(G),\ u = \bar{y}*\check{x}.$$

It follows that the w^*-, weak operator and ultra-weak operator topologies on $W^*(G)$ coincide.

For $u \in A(G)$, $T \in W^*(G)$, define the operator $uT \in W^*(G)$ by

$$\langle uT, v \rangle = \langle T, uv \rangle, \quad v \in A(G).$$

It follows readily that $W^*(G)$ is an $A(G)$-module and that $||uT|| \leqslant ||u||\ ||T||$.

A linear functional m on $W^*(G)$ is called a <u>mean</u> if

(i) $m(T) \geqslant 0$ if $T \geqslant 0$ and

(ii) $m(I) = 1$ where I is the identity operator.

m is called an <u>invariant mean</u> if it satisfies (i), (ii) and

(iii) $m(uT) = m(T)$ for $T \in W^*(G)$, $u \in P$ or equivalently

(iii') $m(uT) = u(e)m(T)$ for $T \in W^*(G)$, $u \in A(G)$.

A linear functional m satisfying (iii) or (iii') will be called <u>invariant</u>.

<u>Theorem 9.1</u>. Let G be a discrete group. Then there exists a unique invariant mean on $W^*(G)$.

Proof. The function δ_e defined by $\delta_e(g) = 1$ if $g = e$ and $\delta_e(g) = 0$ otherwise is in $L_2(G)$ and since $\delta_e * \tilde{\delta}_e = \delta_e$ then δ_e is also in $A(G)$. Define the linear functional m on $W^*(G)$ by

$$m(T) = \langle T, \delta_e \rangle = (T\delta_e, \delta_e).$$

It is immediate that m is a mean. We show that m is invariant.

Let $T \in W^*(G)$, $u \in P$. Since $u(e) = 1$ we have $u\delta_e = \delta_e$ and hence

$$m(uT) = \langle uT, \delta_e \rangle = \langle T, u\delta_e \rangle = \langle T, \delta_e \rangle$$
$$= m(T)$$

and m is invariant.

To prove uniqueness, suppose that m' is any invariant mean on $W^*(G)$. Let $T \in W^*(G)$, $v \in A(G)$. We have

$$\langle \delta_e T, v \rangle = \langle T, v\delta_e \rangle = \langle T, v(e)\delta_e \rangle = v(e)\langle T, \delta_e \rangle$$
$$= v(e)m(T) = \langle m(T)I, v \rangle$$

and hence $\delta_e T = m(T)I$. But then since m' is invariant and $\delta_e \in P$,

$$m'(T) = m'(\delta_e T) = m'(m(T)I) = m(T)$$

which proves uniqueness.

Remark. It is well-known and easy to show that in this case, m is a faithful normal trace on $W^*(G)$ so that $W^*(G)$ is of finite type.

Note that in this case, the invariant mean m is an element of $A(G)$ rather than $[W^*(G)]^*$. That this is characteristic of discrete groups is shown in the following proposition.

Proposition 9.2. Let m be an invariant mean on $W^*(G)$. If $m \in A(G)$ then G is discrete.

Proof. By assumption there exists an element $u \in A(G)$ such that $m(T) = \langle T, u \rangle$ for all $T \in W^*(G)$. We then have by invariance,

$$\langle T, u \rangle = m(T) = m(vT) = \langle vT, u \rangle = \langle T, uv \rangle$$

for all $T \in W^*(G)$, $v \in P$. Hence $u = vu$ for all $v \in P$. Suppose now that G is non-discrete. Then we can choose $g(\neq e)$ such that $u(g) \neq 0$ and $v \in P$ with $v(g) = 0$. But then $u \neq vu$ and this is a contradiction. Hence G is discrete.

We now consider the general locally compact case. As shown in §2, a useful aid to the study of amenability is the concept of convergence to invariance. Here we introduce a similar idea to obtain the existence of invariant means on $W^*(G)$.

Throughout we shall always consider $A(G)$ as a subspace of $W^*(G)^*$.

Definition 9.3. A net $\{u_\alpha\} \subset P$ is w^*(norm) convergent to invariance if for all $v \in P$,

$$w^*\text{-}\lim (u_\alpha v - u_\alpha) = 0 \text{ i.e. } \lim \langle T, u_\alpha v - u_\alpha \rangle = 0$$

for all $T \in W^*(G)$.

$$(\lim \|u_\alpha v - u_\alpha\| = 0).$$

The existence of such nets follows from the following lemmas due to Eymard [17].

Lemma 9.4. Let V be a neighbourhood of e. Then there exists an element $u \in P$ with spt u (support of u) $\subseteq V$.

Proof. Let K be a compact, symmetric neighbourhood of e with $K^2 \subseteq V$. Let $\chi = \chi_K$ be the characteristic function of K. Define $u(g) = ||\chi||_2^{-2}\ \chi*\tilde{\chi}(g)$. It is easy to verify that $u \in P$. Further

$$u(g) = \frac{1}{||\chi||_2^2} \int_G \chi(h)\chi(g^{-1}h)dh$$

$$= \frac{1}{||\chi||_2^2} \int_K \chi(g^{-1}h)dh$$

Now if $g \notin K^2$ then for all $h \in K$ $g^{-1}h \notin K$ (K is symmetric). Hence $u(g) = 0$ for $g \notin V$.

<u>Lemma 9.5</u>. Let $h \in G$, $u \in A(G)$ be such that $u(h) = 0$. Then there exists a sequence $\{u_n\} \subset A(G)$ such that $\lim ||u_n - u|| = 0$ and each u_n vanishes on some neighbourhood of h (varying with n).

For a proof see [17] p.229. The result depends on some deep analysis of A(G).

<u>Lemma 9.6</u>. If K is a compact subset of G, there exists an element $u \in A(G)$ such that $u(g) \equiv 1$ on K.

Proof. Let V be a compact neighbourhood of e and let χ_V, χ_{KV} denote the characteristic functions of V, KV respectively. Define

$$\begin{aligned} u(g) &= |V|^{-1}\chi_{KV}*\tilde{\chi}_V(g) \\ &= |V|^{-1}\int_G \chi_{KV}(h)\chi_V(g^{-1}h)dh \\ &= |V|^{-1}|KV \cap gV|. \end{aligned}$$

Then $u \in A(G)$ and $u(g) \equiv 1$ on K.

We can now prove.

<u>Proposition 9.7</u>. Let $\{V_\alpha\}$ be a neighbourhood basis for e and for each α, choose $u_\alpha \in P$ with spt $u_\alpha \subseteq V_\alpha$. Directed by inclusion, u_α is norm convergent to invariance.

Proof. Fix $v \in P$, $\varepsilon > 0$. Choose $v' \in P$ with compact support K such that $||v - v'|| < \varepsilon/2$. By lemma 9.6 choose $u \in A(G)$ with $u(g) \equiv 1$ on K. Then in particular $(u - v)(e) = 0$ so that by lemma 9.5 there exists some $w \in A(G)$ with $||u - v - w|| < \varepsilon/2$ and w identically zero on some neighbourhood U of e. Then if $V_\alpha \subset U \cap K$ we have $u_\alpha u = u_\alpha$ and $u_\alpha w = 0$. Hence for such α's,

$$\begin{aligned} ||u_\alpha v - u_\alpha|| &\leqslant ||u_\alpha(v - v')|| + ||u_\alpha v' - v_\alpha|| \\ &< \varepsilon/2 + ||u_\alpha v' - u_\alpha|| \\ &= \varepsilon/2 + ||u_\alpha(v' - u - w)|| \\ &\leqslant \varepsilon/2 + ||v' - u - w|| \\ &< \varepsilon \end{aligned}$$

and $\{u_\alpha\}$ is norm convergent to invariance.

With this result we obtain the existence of invariant means on $W^*(G)$.

<u>Theorem 9.8</u>. Let N denote the set of invariant means on $W^*(G)$. Then N is a non-empty convex set.

Proof. By proposition 9.7 there exists a net $\{u_\alpha\} \subset P$ which is w^*-convergent to invariance. Since the unit ball of $W^*(G)^*$ is w^*-compact, $\{u_\alpha\}$ has a w^*-accumulation point m. Let $\{u_\beta\}$ be any subnet, w^*-convergent to m. Each u_β being in P is a mean so that m is a mean. If $u \in P$, $T \in W^*(G)$ then

$$\begin{aligned} m(uT) &= \lim \langle uT, u_\beta \rangle \\ &= \lim \langle T, u_\beta u \rangle \end{aligned}$$

and since $\{u_\beta\}$ is w^*-convergent to invariance,

$$\lim |\langle T, u_\beta u \rangle - \langle T, u_\beta \rangle| = 0$$

i.e.
$$\begin{aligned} m(uT) &= \lim \langle T, u_\beta \rangle \\ &= m(T). \end{aligned}$$

Hence m is an invariant mean and N is non-empty. Clearly N is convex.

We now consider the problem of uniqueness. Specifically we want to show that if G is non-discrete then there is more than one invariant mean on $W^*(G)$. We commence by establishing some properties of the dual space $W^*(G)^*$.

The Arens product on $W^*(G)^* = A(G)^{**}$ makes $W^*(G)^*$ into a Banach algebra (but see §4) and in this case is defined as follows.

For $m \in W^*(G)^*$, $T \in W^*(G)$ define $m \circ T \in W^*(G)$ by

$\langle m \circ T, u \rangle = m(uT)$ for all $u \in A(G)$.

For $m, n \in W^*(G)^*$, define $m \circ n \in W^*(G)^*$ by

$\langle m \circ n, T \rangle = \langle m, n \circ T \rangle$ for all $T \in W^*(G)$.

We have

<u>Proposition 9.9</u>. (i) Let $m \in W^*(G)^*$. Then m is invariant iff $m = v \circ m$ for all $v \in P$.

(ii) If m is invariant then so is $m \circ n$ for all $n \in W^*(G)^*$.

Proof. (i) For $T \in W^*(G)$ we have

$\langle v \circ m, T \rangle = \langle m \circ T, v \rangle = m(vT)$ and the result follows.

(ii) If $v \in P$ then by (i)

$v \circ (m \circ n) = (v \circ m) \circ n = m \circ n$.

Hence by (i) again, $m \circ n$ is invariant.

Let H be a compact, normal subgroup of G. In [17] p.217, Eymard showed that $A(G/H)$ may be identified with the subalgebra of $A(G)$ consisting of all functions which are constant on the cosets of H. The adjoint of this identification is a homomorphism Π of $W^*(G)$ onto $W^*(G/H)$.

Similarly we may identify $L_2(G/H)$ with the subspace M of $L_2(G)$ for all functions constant on the cosets of H. For $T' \in W^*(G/H)$ define the operator $\rho T'$ on $L_2(G)$ by

$$\begin{aligned} \rho T'x &= T'x \quad \text{if} \quad x \in M \\ &= 0 \quad\ \ \text{if} \quad x \in M^{\perp}. \end{aligned}$$

It follows readily that $\rho T' \in W^*(G)$ and that ρ is an isometric embedding of $W^*(G/H)$ into $W^*(G)$.

For $u \in A(G)$, $f(T') = \langle u, \rho T' \rangle$ is an ultra-weakly continuous linear functional on $W^*(G/H)$ so that $f(T')$ $= \langle u', T' \rangle$ for some uniquely defined $u' \in A(G/H)$ (in fact $u' = \rho^* u$).

With these results we can prove

<u>Proposition 9.10</u>. Let H be a compact normal subgroup of G and let m be an invariant mean in $W^*(G)^*$, n' an invariant element in $W^*(G/H)^*$. Let $n = \Pi^* n'$. Then $m \circ n$ is invariant and $||m \circ n|| = ||n'||$.

Proof. By proposition 9.9 (ii) we know already that $m \circ n$ is invariant. Also since

$$||m \circ n|| \leqslant ||m|| \ ||n|| = ||\Pi^* n'|| \leqslant ||n'||$$

it suffices to prove that $||m \circ n|| \geqslant ||n'||$.

Fix $\varepsilon > 0$ and choose $T' \in W^*(G/H)$ such that $||T'|| = 1$ and $|\langle n', T' \rangle| \geqslant ||n'|| - \varepsilon$.

If $u \in A(G)$, $v' \in A(G/H)$ ($\subset A(G)$) then

$$\langle \Pi(u\rho T'), v' \rangle = \langle u\rho T', v' \rangle$$
$$= \langle \rho T', uv' \rangle$$
$$= \langle v'\rho T', u \rangle$$
$$= \langle v'T', u' \rangle$$
$$= \langle u'T', v' \rangle$$

so that $\Pi(u\rho T') = u'T'$. Hence

$$\langle n \circ \rho T', u \rangle = \langle n, u\rho T' \rangle$$
$$= \langle \Pi^* n', u\rho T' \rangle$$
$$= \langle n', \Pi(u\rho T') \rangle$$
$$= \langle n', u'T' \rangle$$
$$= u'(e)\langle n', T' \rangle$$

so that $n \circ \rho T' = \langle n', T' \rangle I$.

Therefore

$$|\langle m \circ n, \rho T' \rangle| = |\langle m, n \circ \rho T' \rangle|$$
$$= |\langle m, \langle n', T' \rangle I \rangle|$$
$$= |\langle n', T' \rangle|$$
$$\geqslant ||n'|| - \varepsilon.$$

i.e. $||m \circ n|| \geqslant ||n'||$.

Corollary 9.11. If H is compact and $W^*(G/H)^*$ admits more than one invariant mean then so does $W^*(G)^*$.

Proof. Let m be an invariant mean in $W^*(G)$. Define $\sigma: W^*(G/H)^* \to W^*(G)^*$ by

$$\sigma(n') = m \circ \Pi^* n'.$$

By the previous proposition, σ is an isometry from the subspace of invariant elements in $W^*(G/H)^*$ into the subspace of invariant elements in $W^*(G)^*$. Consequently if n_1', n_2' are distinct invariant means in $W^*(G/H)^*$ then $\sigma(n_1')$, $\sigma(n_2')$ are distinct invariant means in $W^*(G)^*$.

Finally we shall need

Proposition 9.12. Let K be an open subgroup of G and suppose that there are two distinct invariant means on $W^*(K)$. Then there are two distinct invariant means on $W^*(G)$.

Remark. For G compact, K a closed subgroup this was proved in [11] theorem 7. The proof given below is perhaps a little simpler.

Proof. Let ϕ denote the restriction homomorphic map, $\phi: A(G) \to A(K)$. ϕ is bounded and since K is open, ϕ is onto ([17], p.215). ϕ^* is an isometric embedding of $W^*(K)$ into $W^*(G)$ from which it follows that ϕ^{**} maps $W^*(G)^*$ onto $W^*(K)^*$. Let M denote the set of means in $W^*(G)^*$ and M' the set of means in $W^*(K)^*$. We show firstly that $\phi^{**}(M) = M'$.

Let $m \in M$, $T' \in W^*(K)$ with $T' \geqslant 0$. Then $\phi^*T' \geqslant 0$ and

$$\langle \phi^{**}m, T' \rangle = \langle m, \phi^*T' \rangle \geqslant 0.$$

Further $\langle \phi^{**}m, I\rangle = \langle m, \phi^{*}I\rangle = \langle m, I\rangle = 1$ and therefore $\phi^{**} m$ is a mean i.e. $\phi^{**}(M) \subseteq M'$. Conversely let $m' \in M'$. m' is a linear functional on $W^{*}(K)$ (which maybe considered as a subspace of $W^{*}(G)$ under the isometry ϕ^{*}) satisfying $||m'|| = m'(I) = 1$. By the Hahn-Banach theorem there exists an extension m of m' to all of $W^{*}(G)$ satisfying $||m|| = m(I) = 1$. But then m is a mean and obviously $\phi^{**}m = m'$. Hence $\phi^{**}(M) = M'$.

The proposition will now be proved once we show that ϕ^{**} maps invariant means to invariant means. Let m be an invariant mean on $W^{*}(G)$. Then $\phi^{**} m$ is a mean on $W^{*}(K)$. Note firstly that if $u' \in A(K)$, $T' \in W^{*}(K)$ and if $u \in A(G)$ is such that $\phi(u) = u'$ then $\phi^{*}(u'T') = u\phi^{*}(T')$. [In fact if $v \in A(G)$ then $\langle \phi^{*}(u'T'), v\rangle = \langle u'T', \phi v\rangle =$ $\langle T', u'\phi v\rangle = \langle T', \phi(uv)\rangle = \langle \phi^{*}T', uv\rangle = \langle u\phi^{*}T', v\rangle$].
Now let $u' \in A(K)$, $T' \in W^{*}(K)$. Then

$$\begin{aligned}\langle \phi^{**}m, u'T'\rangle &= \langle m, \phi^{*}(u'T')\rangle \\ &= \langle m, u\phi^{*}T'\rangle \quad \text{if } \phi(u) = u' \\ &= u(e)\langle m, \phi^{*}T'\rangle \\ &= u(e)\langle \phi^{**}m, T'\rangle\end{aligned}$$

i.e. ϕ^{**} is invariant.

We can now prove our main result.

<u>Theorem 9.13</u>. If G is non-discrete, $W^{*}(G)^{*}$ has more than one invariant mean.

Proof. Let C be a compact neighbourhood of e. The subgroup K of G generated by C is an open subgroup and applying proposition 9.12, we may already assume that G is compactly generated. A theorem of Kakutani and Kodaira (see [23], theorem 8.7) now applies and we find a compact normal subgroup H such that G/H has a countable basis for its open sets. If G/H is discrete then H is an open, compact non-discrete subgroup of G and a theorem of Dunkl and Ramirez ([11], theorem 11) shows that W*(K)* has more than one invariant mean. But then by proposition 9.12 again so does W*(G)*. We may therefore assume from the outset that G has a countable basis for its open sets. Assume that there is a unique invariant mean m on W*(G). Let $\{V_n\}_{n=1}^{\infty}$ be a decreasing neighbourhood basis at e. For each n choose $u_n \in P$ with spt $u_n \subseteq V_n$. By proposition 9.7, $\{u_n\}$ is norm-convergent to invariance and by the proof of theorem 9.8, each w*-accumulation point of $\{u_n\}$ in W*(G)* is an invariant mean. It follows by the uniqueness of m that w*-lim $u_n = m$. In other words $\{u_n\}$ is a weak Cauchy sequence in A(G). But A(G) is the predual of the W*-algebra W*(G) so that a theorem of Sakai [37] applies to show that A(G) is weakly sequentially complete. This implies that $m \in A(G)$. Proposition 9.2 now shows that G is discrete and this is a contradiction. Applying theorem

9.8 we see that $W^*(G)^*$ must in fact have at least uncountably many invariant means.

We have already remarked that if G is discrete then the unique invariant mean on $W^*(G)$ is a faithful trace on $W^*(G)$. At the other extreme we have

Theorem 9.14. If G is non-discrete and m is an invariant mean on $W^*(G)$ then $m(T) = 0$ for all $T \in C^*(G)$.

Proof. Since $L_1(G)$ is norm dense in $C^*(G)$ it suffices to show that $m(L_f) = 0$ for all $f \in L_1(G)$.

Let $f \in L_1(G)$, $\varepsilon > 0$. Let U be a neighbourhood of e such that $\int_U |f(g)|dg < \varepsilon$ (G is non-discrete) and choose $u \in P$ with spt $u \subset U$. Choose $\{u_\alpha\} \subset P$ such that $w^*\text{-}\lim u_\alpha = m$. Then $\{u_\alpha\}$ is w^*-convergent to invariance so that

$$\begin{aligned} m(L_f) &= \lim \int_G f(g)u_\alpha(g)dg \\ &= \lim \int_G f(g)u(g)u_\alpha(g)dg. \end{aligned}$$

But
$$\begin{aligned} \left|\int_G f(g)u(g)u_\alpha(g)dg\right| &\leqslant \int_G |f(g)||u(g)||u_\alpha(g)|dg \\ &\leqslant \int_U |f(g)|dg \\ &< \varepsilon \text{ for all } \alpha. \end{aligned}$$

Hence $m(L_f) = 0$.

§10. A GENERAL FORM OF BOCHNER'S THEOREM

A positive definite function on G may be defined as a function $\phi \in L_\infty(G)$ satisfying

$$\langle \phi, f^{*} * f \rangle \geqslant 0 \text{ for all } f \in L_1(G).$$

If G is Abelian then we may dualize this result by defining $T \in L_\infty(\hat{G})$ to be positive definite if $\langle T, u \rangle \geqslant 0$ for all $u \in L_1(\hat{G})$ with Fourier transform $\hat{u}$ numerically positive. Bochner's theorem now states that every positive definite function $T \in L_\infty(\hat{G})$ is determined by a positive measure μ in $M(G)$. In this section we shall extend Bochner's theorem to arbitrary amenable groups thus showing that its validity depends not so much on the theory of Fourier transforms but rather on the existence of approximate identities.

Definition 10.1. Let G be a locally compact group. An operator $T \in W^*(G)$ will be called positive definite if $\langle T, u \rangle \geqslant 0$ for all $u \in A(G)$ satisfying $u(g) \geqslant 0$ for all $g \in G$.

We shall prove

Theorem 10.2. If G is amenable then there exists a 1-1 and onto correspondence between positive definite operators T and positive measures $\mu \in M(G)$ defined by

$T \leftrightarrow L_\mu$ where $L_\mu x = \mu * x$, $x \in L_2(G)$. Moreover $||T|| = ||\mu||$.

The proof requires two preliminaries results.

Lemma 10.3. If G is amenable then A(G) has an approximate identity.

Proof. (Leptin [28]). For K a compact set, $e \in K$ and $0 < \varepsilon < 1$, choose U compact with $|U| > 0$ and $|U|^{-1}|KU| \leqslant (1-\varepsilon)^{-2}$ (c.f. theorem 3.1). Let $\phi = \chi_{KU}$, $\psi = \chi_U$ and put $e_{K,\varepsilon} = (1-\varepsilon)|U|^{-1}\phi * \tilde{\psi}$. We have $e_{K,\varepsilon} \in A(G)$, $e_{K,\varepsilon}(g) = 1-\varepsilon$ if $g \in K$ and

$$||e_{K,\varepsilon}|| = |U|^{-1}(1-\varepsilon)||\phi * \tilde{\psi}|| \leqslant (1-\varepsilon)|U|^{-1}||\phi||_2 ||\psi||_2$$
$$\leqslant (1-\varepsilon)|U|^{-1}[|KU||U|]^{\frac{1}{2}} \leqslant 1.$$

Define $e_{K,\varepsilon} \leqslant e_{L,\eta}$ if $K \supseteq L$, $\varepsilon \leqslant \eta$. Clearly $\{e_{K,\varepsilon}\}$ is a net in A(G). We show that it is an approximate identity.

Let $u \in A(G)$, $\varepsilon > 0$. Since $C_{00}(G) \cap A(G)$ is dense in A(G), choose $v(\neq 0) \in A(G) \cap C_{00}(G)$ with compact support K_0 such that $||u-v|| < \varepsilon/4$. Then for all compact $K \supseteq K_0$, $\eta < \varepsilon/(2||v||)$ we have

$$||e_{K,\eta}u - u|| \leqslant ||e_{K,\eta}(u-v)|| + ||e_{K,\eta}v - v|| + ||v-u||$$
$$\leqslant 2\varepsilon/4 + ||e_{K,\eta}v - v||$$
$$= \varepsilon/2 + ||(1-\eta)v - v||$$

$$= \varepsilon/2 + \eta||v||$$
$$< \varepsilon.$$

Hence $\lim e_{K,\varepsilon}u = u$ which proves the lemma.

Lemma 10.4. If A is a Banach*algebra with approximate identity and B the enveloping C*-algebra of A then the restriction map is a 1-1 and onto norm preserving map of the positive functionals of A onto the positive functionals of B.

This is a well known result, see e.g. Dixmier [6] prop. 2.7.5.

Proof of theorem 10.2. A(G) is a Banach*algebra under conjugation and since G is amenable, A(G) has an approximate identity by lemma 10.3. If now T is a positive definite operator in W*(G) then $f(u) = \langle T, u\rangle$ extends by lemma 10.4 to a positive functional $\tilde{f}$ on $C_0(G)$ with $||\tilde{f}|| = ||f||$. Hence there exists a positive radon measure $\mu \in M(G)$ such that $\tilde{f}(v) = \int_G v(g)d\mu(g)$ for all $v \in C_0(G)$. In particular if $v \in A(G)$ with $v = \bar{y} * \check{x}$ then

$$\begin{aligned} \langle Tx, y\rangle &= \langle T, v\rangle \\ &= \int_G v(g)d\mu(g) \\ &= \int_G\int_G \bar{y}(h)\check{x}(h^{-1}g)dhd\mu(g) \end{aligned}$$

$$= \int_G [\int_G x(g^{-1}h)d\mu(g)]\bar{y}(h)dh$$

$$= (\mu * x,\ y)$$

$$= (L_\mu x,\ y)$$

Hence $T = L_\mu$. Moreover since μ is a positive measure and G is amenable, a result of Gilbert [19] implies that $||\mu|| = ||L_\mu||$. This proves the theorem.

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